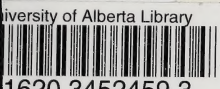


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MODULE 1

Precalculus



MATHEMATICS



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Mathematics 31

Module 1

PRECALCULUS



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This document is intended for	
Students	✓
Teachers (Mathematics 31)	✓
Administrators	
Parents	
General Public	
Other	

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 Student Module Booklet
 Module 1
 Precalculus
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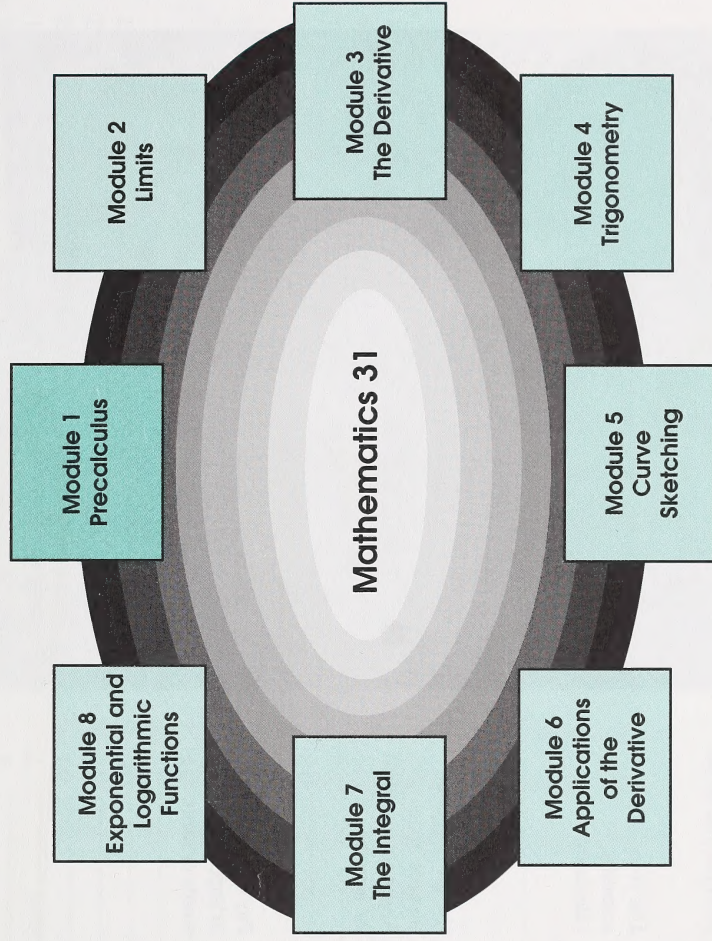
Welcome



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Welcome to Module 1. We hope you'll enjoy your study of Precalculus.

Mathematics 31 contains eight modules. Work through modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.



- Use your graphing calculator.



- Use your scientific calculator.



- Use computer software.



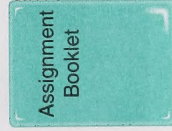
- Use the suggested answers in the Appendix to correct the activities.



- View a videocassette.



- Pay close attention to important words or ideas.



- Answer the questions in the assignment booklet



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.

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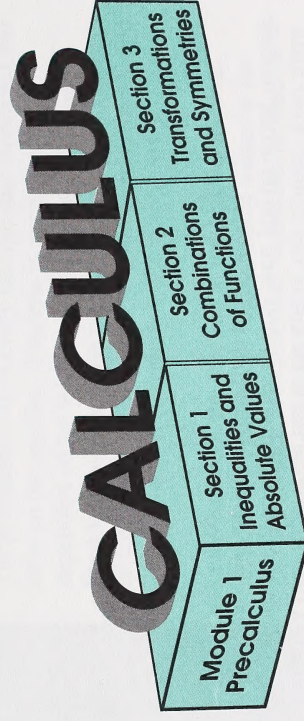
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Module Overview

When building a home, many people would not consider constructing on anything but a solid foundation. The durability of the building itself depends on the care with which the foundation is laid. The stronger the foundation, the larger the structure can be (safely supported). And if, because of future demand, additions to the house are required, the homeowner will be able to rest easily, knowing that the foundation is sound enough to support any addition.

Calculus can only be built on a solid foundation. A foundation of concepts and procedures, many of which you have already studied in previous mathematics courses—in particular Mathematics 20. The algebra of polynomials, factoring, operations with rational expressions, exponents and radicals, and equation solving are a few of the skills you will need in Mathematics 31. In this module you will acquire additional skills essential to the study of calculus.

In Section 1 you will review the fundamentals of inequalities and absolute value. Throughout this section, a notation will be used to express your answers, which may be new to you, called interval notation. In Section 2 you will be introduced to the algebra of functions. Finally, Section 3 builds on Section 2 and deals specifically with analysing and transforming the graphs of functions.



Evaluation

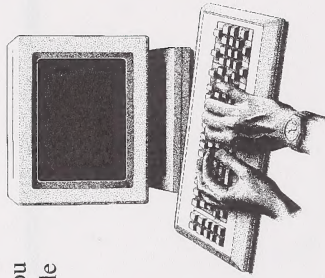
Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete three section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	38 marks
Section 2 Assignment	26 marks
Section 3 Assignment	26 marks
Final Module Assignment	10 marks
TOTAL	100 marks

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.



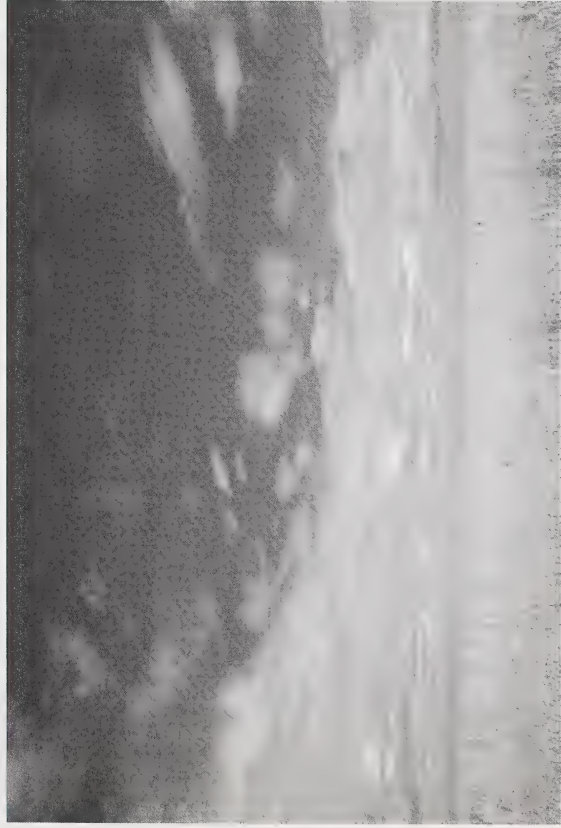
If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Inequalities and Absolute Values



The weather on the Canadian Prairies is one of extremes: extremes of temperature, wind, rain, or snow. These extremes affect every lifestyle and livelihood that exist on the prairies. Farmers, in particular, are dependent on the weather. In the spring, for instance, knowledge of long-term averages and what the extremes of temperatures can be govern when crops are sown and what varieties are planted.

In mathematics, the solution of inequalities involves determining extremes as well—finding values of the variables which, within certain constraints or specific intervals, will satisfy the original conditions.

This section consists of three activities. In Activity 1, you will be introduced to interval notation and how to express an inequality using this notation (how to illustrate its graph on the real number line). In Activities 2 and 3, you will study techniques for solving inequalities involving absolute value and quadratic expressions.

Activity 1: Interval Notation



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Suppose you decide to cycle for an hour along a straight section of road that runs past your home. If you know you can average 10 km/h, how far can you cycle in either direction and return home within one hour? You can model the problem using a real number line, locating your home at the origin and representing each kilometre to be cycled as one unit.



It is possible to travel as far as 5 km from your home, in either direction, within one hour. Therefore, to represent the section of road travelled, with x as the directed distance from your home, you can write $-5 \leq x \leq 5$.

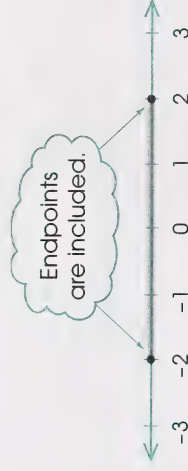
This inequality may be represented in **interval notation** as $[-5, 5]$.



An **interval of real numbers** is the set of real numbers between two stated values called the **endpoints** of the interval. One, both, or neither of the two endpoints may be included in the interval.

Example 1

Express the following using an inequality and interval notation.



Solution

Because the endpoints are included, as indicated on the number line, the inequality is written $-2 \leq x \leq 2$; and the interval is represented using square brackets, $[-2, 2]$. Because the interval represents a set of real numbers, when using an inequality, the inequality is often written as part of **set-builder notation** $\{x \mid -2 \leq x \leq 2\}$. This may be read as **the set of all real numbers x , such that x is between -2 and 2 inclusive**.



An interval which contains both its endpoints is called a **closed interval**.



An interval which does not include its endpoints is called an **open interval**.

1. Sketch the closed interval $[-3, 5]$ on a number line. Express this set of real numbers using an inequality.

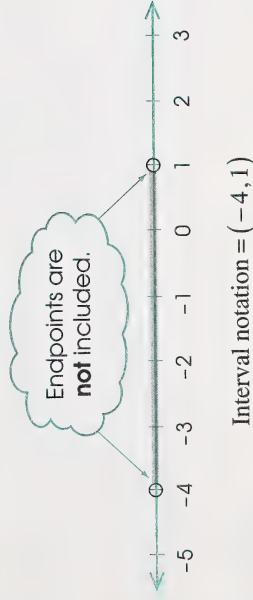


Check your answers by turning to the Appendix.

Example 2

Graph $\{x | -4 < x < 1\}$, and express it in interval notation.

Solution



When the interval does not include its endpoints, round brackets are used. This may be read as **the set of real numbers between -4 and 1 non-inclusive**.

2. Graph and write the open interval $(2, 3)$ as an inequality.



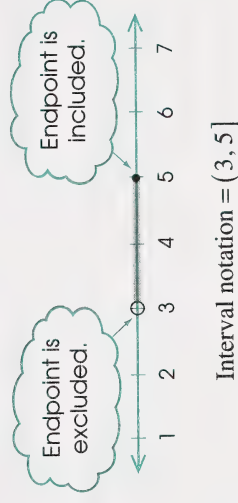
Check your answers by turning to the Appendix.

Sometimes you will see that the interval includes only one of its endpoints.

Example 3

Write $\{x | 3 < x \leq 5\}$ in interval notation, and graph this set on the number line.

Solution



3. Write $[-3, 0)$ in set notation using an inequality. Use your own words to describe this set.



Check your answers by turning to the Appendix.

You will need to consider **infinite intervals** as well. Recall that the symbol for infinity is ∞ .

Example 4

Describe $(4, \infty)$ in set-builder notation.

Solution

$$\{x | x > 4\}$$



Example 5

Define the following graph using an inequality and using interval notation.



Solution

$$\{x | x \leq 3\} \text{ or } (-\infty, 3]$$

When you use the symbols ∞ and $-\infty$ in interval notation, you must not think of these numbers like 3 or -4 , but rather that the graphs extend to the right or to the left without bound.

Example 6

What does $(-\infty, \infty)$ represent?

Solution

The set of real numbers; its graph would be the entire number line.

4. Illustrate each interval graphically.

a. $(3, \infty)$ or $x > 3$

b. $(-\infty, 6)$ or $x < 6$

c. $[4, \infty)$ or $x \geq 4$

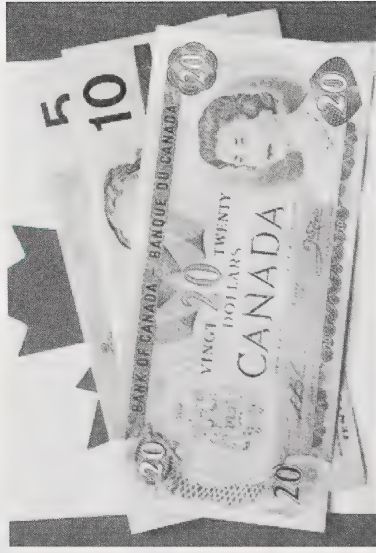


Check your answers by turning to the Appendix.

Interval notation is used to denote solutions to inequalities, such as the cycling example at the beginning of this activity.

Activity 2: Inequalities Involving Absolute Values

Have you ever owed someone money?



If you owed a friend \$5.00, you wouldn't normally think *minus* \$5.00; that is, you wouldn't think of it as a signed number. Instead, you would probably think of the \$5.00 you must earn to repay your friend. When only size or magnitude of a number is to be considered, absolute value is often used to represent that concept. You may recall that the absolute value of a real number is non-negative.

For instance, $|-3| = 3$, $|5| = 5$, and $|0| = 0$.



Therefore, **absolute value of a real number** is defined as follows:

If $x \geq 0$, then $|x| = x$; and if $x < 0$, then $|x| = -x$.

Notice that if $x < 0$, its absolute value is the opposite of x .

Example 1

Use the definition of absolute value to find $|-6|$.

Solution

Because $-6 < 0$, you will use $|x| = -x$. Note that

$$|-6| = -(-6) = 6$$

The opposite of -6 .

Example 2

Graph $|x| \leq 2$.

Solution

In this example, you must think of real numbers that have absolute values less than or equal to 2. A few representative values are $|-1| = 1$, $|-1.7| = 1.7$, $|1.6| = 1.6$, and $|2| = 2$. On a number line, note that the appropriate interval is as follows:



You may write the solution in set notation as $\{x | -2 \leq x \leq 2\}$ or in interval notation as $[-2, 2]$.

Try the following questions.

1. Solve $|x| = 4$.
2. Solve and graph $|x| < 4$.



Check your answers by turning to the Appendix.

Example 3

Solve $|3 + 2x| = 7$.

Solution

$$\begin{aligned} 3 + 2x &= 7 \quad \text{or} \quad 3 + 2x = -7 \\ 2x &= 4 & 2x &= -10 \\ x &= 2 & x &= -5 \end{aligned}$$

The solution set is $\{-5, 2\}$.

Example 4

Solve $|3 + 2x| \leq 7$.

Solution

Method 1

If you look at Example 3, the values of x for which $|3 + 2x| = 7$ were -5 and 2 . Therefore, the values of x for which $|3 + 2x| \leq 7$ must lie between -5 and 2 .



Use a calculator to test values from the interval $[-5, 2]$. Do they satisfy the original condition? Do values beyond this interval satisfy the original condition?

Method 2

$|3 + 2x| \leq 7$ means $-7 \leq 3 + 2x \leq 7$.

$$\therefore -7 - 3 \leq 3 - 3 + 2x \leq 7 - 3$$

$$-10 \leq 2x \leq 4$$

$$\frac{-10}{2} \leq \frac{2x}{2} \leq \frac{4}{2}$$

$$-5 \leq x \leq 2$$

Therefore, the solution is the interval $[-5, 2]$.

3. Solve $|4 - x| < 7$.



Check your answers by turning to the Appendix.

Another class of inequalities are those of the form $|f(x)| \geq a$.

Example 5

Solve $|x| > 7$.

Solution

Because the magnitude of x must exceed 7, $x > 7$ or $x < -7$.



Two disjoint intervals on the number line contain solutions to this inequality.

The solution set is $\{x | x < -7 \text{ or } x > 7\}$, or, in interval notation, $(-\infty, -7) \cup (7, \infty)$.



The symbol $\cup$ stands for the **union of the two sets**. That is, both segments must be considered. Solutions lie in either of the intervals.

Example 6

Solve $|4 - 3x| > 7$.

Solution

$|4 - 3x| > 7$ means

$$\begin{array}{lll} 4 - 3x > 7 & \text{or} & 4 - 3x < -7 \\ -3x > 3 & & -3x < -11 \\ -\frac{1}{3}(-3x) < 3\left(-\frac{1}{3}\right) & & -\frac{1}{3}(-3x) > -11\left(-\frac{1}{3}\right) \\ x < -1 & & x > \frac{11}{3} \end{array}$$

Multiplying or dividing by a negative value reverses the inequality.

The solution set is $\{x | x < -1 \text{ or } x > \frac{11}{3}\}$ or $(-\infty, -1) \cup (\frac{11}{3}, \infty)$.



Use your calculator to check the intervals against the original condition.

Example 7

$$\text{Solve } \frac{x-3}{x-4} \geq 2.$$

Solution

To eliminate the denominator from the fractional expression, multiply both sides by $(x-4)$. Since dividing by zero is undefined, you know $x-4 \neq 0$. Thus, there are two possibilities: $x-4 > 0$ or $x-4 < 0$.

Case 1: Assume $x-4 > 0$.

$$\begin{aligned}\frac{x-3}{x-4}(x-4) &\geq 2(x-4) \\ x-3 &\geq 2x-8 \\ -x &\geq -5 \\ x &\leq 5\end{aligned}$$

You must be careful to reconcile your assumption that $x-4 > 0$ or $x > 4$ with $x \leq 5$. Therefore, the interval $(4, 5]$ contains solutions to the inequality.

Case 2: Assume $x-4 < 0$.

This time multiplying by $x-4$ yields the following:

$$\begin{aligned}\frac{x-3}{x-4}(x-4) &\leq 2(x-4) \\ x-3 &\leq 2x-8 \\ -x &\leq -5 \\ x &\geq 5\end{aligned}$$

Because your assumption, $x-4 < 0$ or $x < 4$, contradicts the statement $x \geq 5$, Case 2 does not yield a solution. Thus, the only solution is from Case 1.

Therefore, the solution set is $\{x | 4 < x \leq 5\}$.

Try the following question. Model your solution to Example 7.

4. Solve $\frac{x-3}{x+5} \leq 3$.



Check your answers by turning to the Appendix.

Example 8

Solve and graph $\frac{|x-3|}{|x-4|} > 6$.

Solution

$$\frac{|x-3|}{|x-4|} > 6 \quad \textcircled{1} \quad |x-3| > 6|x-4| \quad \textcircled{2}$$

Because you are dealing with absolute value, you are not certain whether the expressions within the absolute value symbols are positive or negative.

Possible Combinations		
	$x-3$	$x-4$
Case 1	+	+
Case 2	+	-
Case 3	-	+
Case 4	-	-

Why can't either binomial equal zero? If $x-3=0$, then $0 > 6$ when substituted into the original inequality. If $x-4=0$, the fractional expression in the original inequality would be undefined.

Case 1: $x-3 > 0$ and $x-4 > 0$. For both to be positive, $x > 4$. You will search for solutions in the interval $(4, \infty)$.

According to the definition of absolute value, because both binomials are positive, $|x-3| = x-3$ and $|x-4| = x-4$.

Therefore, substitute in $\textcircled{2}$.

$$\begin{aligned} (x-3) &> 6(x-4) \\ x-3 &> 6x-24 \\ -5x &> -21 \\ x &< 4\frac{1}{5} \end{aligned}$$

Since, you are searching for solutions in $(4, \infty)$, the interval obtained from this case is $(4, 4\frac{1}{5})$.

Case 2: $x-3 > 0$ and $x-4 < 0$. This implies $x > 3$ and $x < 4$. This time you will search for additional solutions in the interval $(3, 4)$. As before, $|x-3| = x-3$; but $|x-4| = -(x-4)$ because $x-4$ is negative.

Therefore, substitute in $\textcircled{2}$.

$$\begin{aligned} (x-3) &> -6(x-4) \\ x-3 &> -6x+24 \\ 7x &> 27 \\ x &> 3\frac{6}{7} \end{aligned}$$

Since you are searching for solutions in $(3, 4)$, the interval obtained from this case is $(3\frac{6}{7}, 4)$.

Case 3: $x - 3 < 0$ and $x - 4 > 0$. This implies $x < 3$ and $x > 4$. Impossible! Go to Case 4.

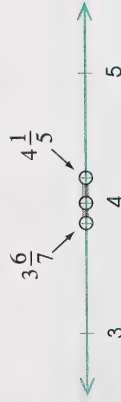
Case 4: $x - 3 < 0$ and $x - 4 < 0$. For both to be negative, $x < 3$. This time you will search for additional solutions in the interval $(-\infty, 3)$. Because both binomials are negative, $|x - 3| = -(x - 3)$ and $|x - 4| = -(x - 4)$.

Therefore, substitute in ②.

$$\begin{aligned} -(x - 3) &> -6(x - 4) \\ -x + 3 &> -6x + 24 \\ 5x &> 21 \\ x &> 4\frac{1}{5} \end{aligned}$$

Since you are searching for solutions in $(-\infty, 3)$, no solution is obtained from Case 4.

Therefore, combining the solutions from Case 1 and Case 2 you obtain $(3\frac{6}{7}, 4) \cup (4, 4\frac{1}{5})$ or $\{x | 3\frac{6}{7} < x < 4 \text{ or } 4 < x < 4\frac{1}{5}\}$.



5. Solve $\frac{|x - 3|}{x + 2} \leq 2$.

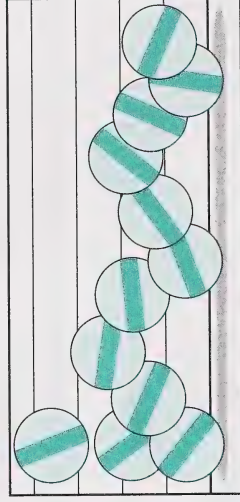


Check your answers by turning to the Appendix.

Real-world contexts that require magnitude only, such as *amount of money* or *distance travelled*, often involve absolute value.

Activity 3: Quadratic Inequalities

Have you ever bounced a ball and wondered how long the ball remains in the air?

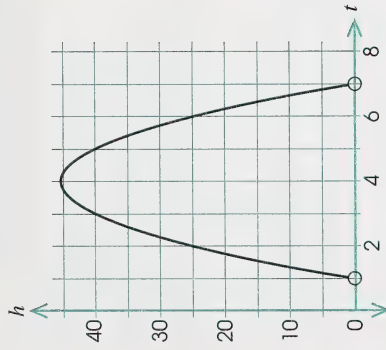


A ball is hurled straight downward striking the ground one second after being released. If that ball bounces back up at a speed of 30 m/s, its height h (in metres) after time t (in seconds) can be approximated by $h = -5t^2 + 40t - 35$, where $t \geq 1$. When the ball is above the ground, $h > 0$. Therefore, to determine the time interval in question, you must solve $-5t^2 + 40t - 35 > 0$.



This is an example of a **quadratic inequality**.

To solve $-5t^2 + 40t - 35 > 0$, you could graph the function $h = -5t^2 + 40t - 35$ and determine the time interval for which the function has a positive value.



From the graph, you see $h > 0$ when $1 < t < 7$; that is, the ball is in the air between $t = 1$ s and $t = 7$ s.

Are there alternate methods for solving quadratic inequalities?

Example 1

Solve $-5t^2 + 40t - 35 > 0$.

Solution

$$-5t^2 + 40t - 35 > 0$$

$$-5(t^2 - 8t + 7) > 0$$

$$t^2 - 8t + 7 < 0$$

$$(t-1)(t-7) < 0$$

When an inequality is divided by a negative value, change the direction of the inequality.

This inequality may be solved by noting that if the product of the two factors is negative, then the factors must differ in sign.

Possible Combinations	
$t-1$	$t-7$
Case 1 +	-
Case 2 -	+

Case 1: If $t-1 > 0$ and $t-7 < 0$, then $t > 1$ and $t < 7$. This yields the same interval $(1, 7)$ that you obtained graphically.

Case 2: If $t-1 < 0$ and $t-7 > 0$, then $t < 1$ and $t > 7$. This is impossible!

Example 2

Solve $x^2 \geq 3x + 4$.

Solution

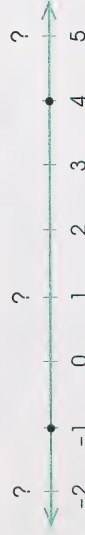
First, you must rewrite the inequality so that the right side is zero and the quadratic is in standard form ($ax^2 + bx + c$).

$$\begin{aligned}x^2 - 3x - 4 &\geq 0 \\(x - 4)(x + 1) &\geq 0\end{aligned}$$

You may solve this condition in two stages.

$$\text{First } (x - 4)(x + 1) = 0 \text{ or } (x - 4)(x + 1) > 0.$$

From the equation, $x = 4$ or $x = -1$. If these roots are graphed, they partition the number line into three intervals where $(x - 4)(x + 1)$ is either positive or negative.



For $(x - 4)(x + 1) > 0$, there are two possibilities.

Possible Combinations		
	$x - 4$	$x + 1$
Case 1	+	+
Case 2	-	-

Case 1: $x - 4 > 0$ and $x + 1 > 0$
 $x > 4$ $x > -1$

Since both must be positive, $x > 4$ and $(4, \infty)$ is part of the solution set.

Case 2: $x - 4 < 0$ and $x + 1 < 0$
 $x < 4$ $x < -1$

Since both must be negative, $x < -1$ and $(-\infty, -1)$ is part of the solution set. If you combine the results from the equation and inequality, the solution is $(-\infty, -1] \cup [4, \infty)$.

Since -1 and 4 were obtained from the equation, they were included in the answer.

You are ready to practise these procedures.

1. Solve the inequality $x^2 - 3x + 2 < 0$ by sketching $y = x^2 - 3x + 2$ and determining the interval for which $y < 0$.
2. Solve $2x^2 > 5x - 2$.



Check your answers by turning to the Appendix.

The first two examples dealt with quadratic expressions which could be readily factored; but what about inequalities such as

$$x^2 - 2x - 2 \leq 0?$$

Example 3

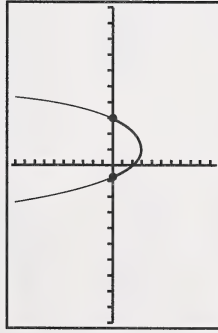
$$\text{Solve } x^2 - 2x - 2 \leq 0.$$

Solution

Method 1



Using a graphing calculator, graph $y = x^2 - 2x - 2$ and use the Trace function to approximate the endpoints of the interval for which $y \leq 0$. These are the x -intercepts.



Did you find that the approximate solution is $-0.7 \leq x \leq 2.7$?

Method 2

Using the quadratic formula, you can determine the exact location of the x -intercepts.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this instance $a = 1$, $b = -2$, and $c = -2$.

$$\begin{aligned} \therefore x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{12}}{2} \\ &= \frac{2 \pm 2\sqrt{3}}{2} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

Because the inequality reads **less than or equal to 0**, these values are members of the solution set. If you graph $1 \pm \sqrt{3}$ on the number line, you will partition the number line into three intervals for which the expression $f(x) = x^2 - 2x - 2$ is either positive or negative.

Simply randomly select values from each interval and evaluate $x^2 - 2x - 2$.

For instance:



For Interval 1, try $x = -3$.

$$\begin{aligned}f(-3) &= (-3)^2 - 2(-3) - 2 \\&= 13\end{aligned}$$

For Interval 2, try $x = 0$.

$$\begin{aligned}f(0) &= (0)^2 - 2(0) - 2 \\&= -2\end{aligned}$$

For Interval 3, try $x = 4$.

$$\begin{aligned}f(4) &= (4)^2 - 2(4) - 2 \\&= 6\end{aligned}$$

Because Interval 2 resulted in a negative value for $x^2 - 2x - 2$, the graph is as follows:



The solution may be written as $\{x | 1 - \sqrt{3} \leq x \leq 1 + \sqrt{3}\}$ or $[1 - \sqrt{3}, 1 + \sqrt{3}]$.

3. Solve $4x^2 > 2x + 1$.



Check your answers by turning to the Appendix.

Problems involving gravitational forces, such as a bouncing ball, commonly involve quadratics.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended you do the Enrichment. You may decide to do both.

Extra Help

In Activity 2 you learned techniques for solving inequalities. One approach for solving inequalities like $\frac{2x-1}{x+1} \geq 4$ is to transform the inequality so the right side is zero, and the left side is a single fractional expression.

$$\begin{aligned}\frac{2x-1}{x+1} &\geq 4 \\ \frac{2x-1}{x+1} - 4 &\geq 4 - 4 \\ \frac{2x-1-4(x+1)}{x+1} &\geq 0 \\ \frac{-2x-5}{x+1} &\geq 0\end{aligned}$$

This inequality consists of two conditions: $\frac{-2x-5}{x+1} = 0$ and $\frac{-2x-5}{x+1} > 0$.

The first is easily solved. If $-2x - 5 = 0$, then $x = -2.5$. The second is solved by noting that the fractional expression is positive. There are two possibilities shown in the following chart.

Possible Combinations		
	$-2x - 5$	$x + 1$
Case 1	+	+
Case 2	-	-

- Case 1:** $-2x - 5 > 0$ and $x + 1 > 0$ implies that $x < -2.5$ and $x > -1$. Impossible! Go to Case 2.
- Case 2:** $-2x - 5 < 0$ and $x + 1 < 0$ implies that $x > -2.5$ and $x < -1$.
- This yields $(-2.5, -1)$. If this is combined with the fact that $x = -2.5$, then the solution is $[-2.5, -1)$.

Use this method to do any two of the following inequalities.

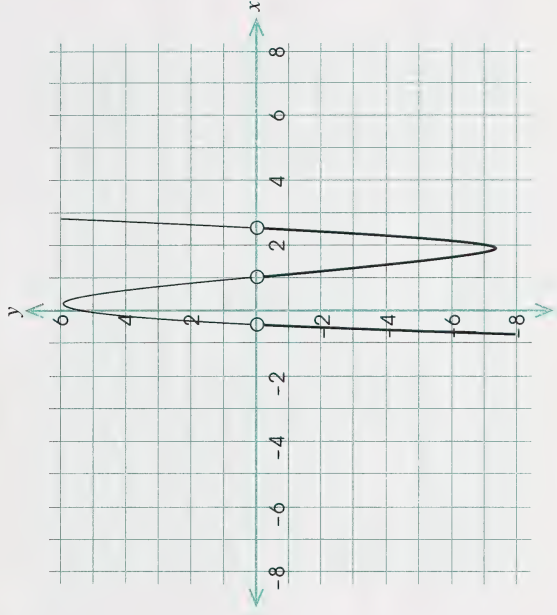
1. Solve $\frac{x}{x-1} > 0$.
2. Solve $\frac{x}{x-1} \leq 2$.
3. Solve $\frac{x+1}{x-1} > -2$.



Check your answers by turning to the Appendix.

Enrichment

You will discover that being able to solve inequalities is an important skill in calculus. In Activity 3 you were introduced to quadratic inequalities. The techniques you learned there may be applied to higher-order polynomial inequalities such as $6x^3 - 19x^2 + 8x + 5 < 0$. If you use a graphing calculator to sketch $y = 6x^3 - 19x^2 + 8x + 5$, you can often determine one or more of the x -intercepts, and at least estimate the intervals of x for which $y < 0$.



Using the Trace function, it appears that an x -intercept of the graph is 1.

This suggests that one of the factors of the cubic is $x - 1$. Use long division or synthetic division to verify this and to find a second factor.

$$\begin{aligned}(6x^3 - 19x^2 + 8x + 5) \div (x - 1) &= 6x^2 - 13x - 5 \\ 6x^3 - 19x^2 + 8x + 5 &= (x - 1)(6x^2 - 13x - 5) \\ &= (x - 1)(3x + 1)(2x - 5)\end{aligned}$$

The complete list of x -intercepts is 1 , $-\frac{1}{3}$, and $\frac{5}{2}$. Using the graph to see where $y < 0$, the solution set of the original inequality is $(-\infty, -\frac{1}{3}) \cup (1, \frac{5}{2})$.



Solve at least two of the following inequalities.

- $x(x - 3)(x + 3) \leq 0$
- $x^3 > 5x^2 - 6x$
- $4x^3 + 4x^2 - x < 1$
- $x^4 - 5x^3 + 5x^2 + 5x - 6 < 0$



Check your answers by turning to the Appendix.



You are now ready to complete the section assignment.

Assignment

Conclusion

You should now be able to write and interpret the solutions to inequalities using any one of the following forms: set-builder notation, a graph, and interval notation.

Also, you should be able to solve inequalities of the following types:

- $|P(x)| \geq a$ or $|P(x)| \leq a$
- $\frac{P(x)}{Q(x)} \geq a$ or $\frac{P(x)}{Q(x)} \leq a$
- $ax^2 + bx + c \geq d$

Inequalities are statements that involve the consideration of a last possible value, or a largest possible value, or both. These considerations are not unlike those of farmers planning when to sow or harvest. Farmers often refer to almanacs to give them facts about the weather—what the long term averages are, and what the temperature or moisture extremes are. Problems involving extremes, like the weather, are common topics in calculus.

Section 2: Combinations of Functions



A remarkable feature of the human mind is its ability to abstract from experience. Geometry, for instance, evolved from people recognizing similarities within their world. Shapes, patterns, and regularities of everyday objects were used to classify and make sense of the environment. Gradually, fundamental concepts such as point, line, and angles were combined with certain assumptions and operations to build an abstract system that could be used to model more complex situations.

One of the fundamental concepts in calculus is the notation of function. A function is simply a relationship between two quantities such as distance travelled and speed or the cost of an article and the tax charged. Functions can be treated as elements, just like points or lines, and can be combined to form abstract structures.

In this section you will study how to combine functions. You will also be shown how to express the sum, difference, product, quotient, and composition of any two functions algebraically and graphically.

Activity 1: Arithmetic Combinations

What do Mandarin and English have in common? They are both languages with words and ways of combining those words to form complex structures (such as sentences) to convey meaning. Just as you can often learn a second language more easily by studying the familiar aspects of your own, in mathematics you can often learn more about an abstract system by studying a related, but more familiar, system. You will discover that the arithmetic of functions is similar to the arithmetic of everyday numbers.

If f and g are functions, then their **sum**, **difference**, **product**, and **quotient** are defined as follows:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x), \\ (f-g)(x) &= f(x) - g(x), \quad (fg)(x) = f(x)g(x), \\ \text{and } \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\end{aligned}$$



Example 1

Given $f(x) = x^2$ and $g(x) = x - 1$, find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$. Show that your answers are correct for $x = 3$.

Solution

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= x^2 + (x-1) \\ &= x^2 + x - 1\end{aligned}$$

$$\begin{aligned}\therefore (f+g)(3) &= 3^2 + 3 - 1 \\ &= 11\end{aligned}$$

Also, since $f(3) = 9$ and $g(3) = 2$, then $f(3) + g(3) = 9 + 2 = 11$.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= x^2 - (x-1) \\ &= x^2 - x + 1\end{aligned}$$

$$\begin{aligned}\therefore (f-g)(3) &= 3^2 - 3 + 1 \quad \text{or} \quad f(3) - g(3) = 9 - 2 \\ &= 7\end{aligned}$$

$$\begin{aligned}(fg)(x) &= f(x)g(x) \\ &= x^2(x-1) \\ &= x^3 - x^2\end{aligned}$$

$$\begin{aligned}\therefore (fg)(3) &= 3^3 - 3^2 \quad \text{or} \quad f(3)g(3) = 9(2) \\ &= 18\end{aligned}$$

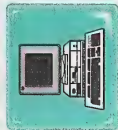
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{x-1}$$

$$\therefore \frac{f(3)}{g(3)} = \frac{3^2}{3-1} \text{ or } \frac{f(3)}{g(3)} = \frac{9}{2}$$

How are the graphs of the sum and difference of two functions related to the original functions?

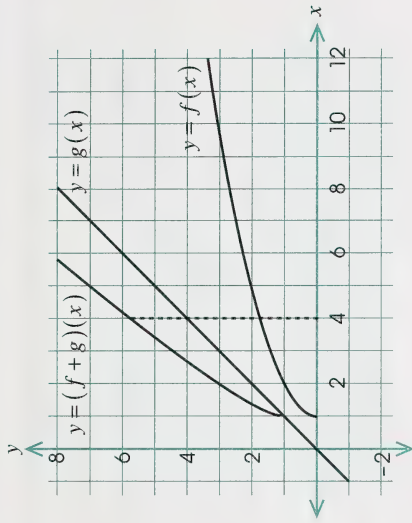
Example 2

Using a graphing calculator or graphing software, graph $y = (f + g)(x)$ and $y = (f - g)(x)$, given $f(x) = \sqrt{x-1}$ and $g(x) = x$. Describe how the graphs are related and how the domains of these functions are related.



Solution

The graph of $y = (f + g)(x)$ can be viewed as being obtained by adding the y-coordinates of the original functions for each value of x . If both ordinates are positive, this is just like laying two “sticks” end to end to obtain a new and longer “stick”.



$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= \sqrt{x-1} + x\end{aligned}$$

Notice that the domain of $f(x) = \sqrt{x-1}$ is $\{x | x \geq 1\}$, since the square roots of negative values are non-real. Even though the domain of $y = g(x)$ is the entire set of real numbers, the domain of $y = (f + g)(x)$ is also $\{x | x \geq 1\}$.



The rule for finding the domain of a sum function is to determine the **intersection of two sets**—the domains of the constituent functions. The intersection of these two sets is the “overlap” of the intervals of the x-axis where the component functions are defined. Thus, $\{x | x \geq 1\} \cap \{\text{Reals}\} = \{x | x \geq 1\}$.

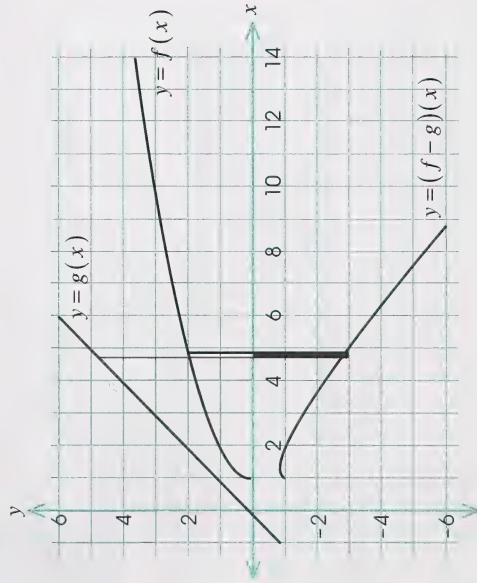
$\cap$ means **intersect**.

The reasons for making the preceding comments can be readily seen by establishing a table of values.

x	-1	0	1	2	3
$f(x)$	undefined	undefined	0	1	$\sqrt{2}$
$g(x)$	-1	0	1	2	3
$(f+g)(x)$	undefined	undefined	1	3	$3+\sqrt{2}$



To find the difference between two functions, the process is similar to finding the sum except that the y -coordinates of the original functions are subtracted.



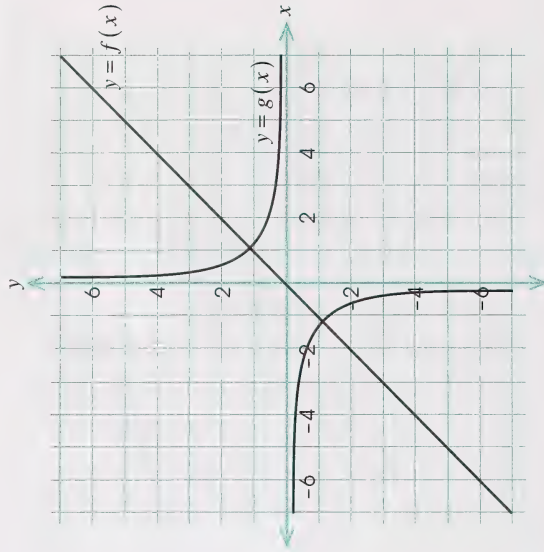
As for the sum, domain of $(f+g) = \text{domain of } f \cap \text{domain of } g$.

Study the following table of values.

x	-1	0	1	2	3
$f(x)$	undefined	undefined	0	1	$\sqrt{2}$
$g(x)$	-1	0	1	2	3
$(f-g)(x)$	undefined	undefined	-1	-1	$\sqrt{2}-3$

You are now ready to do the following questions.

1. Sketch $(f+g)(x)$ and $(f-g)(x)$ for the following functions.



2. Express the sum, difference, product, and quotient of $f(x) = x^2$ and $g(x) = x - 1$ as functions. State the domain of each function.

3. Graph $y = (f + g)(x)$ and $y = (f - g)(x)$ from question 2.



Check your answers by turning to the Appendix.

You discovered in question 2 that the domain of $\frac{f}{g}$ must take into account any values of x for which $g(x) = 0$.



Since division by zero is undefined, you must exclude those values of x from the domain of $\frac{f}{g}$.

Example 3

Determine the domain of $y = \left(\frac{f}{g}\right)(x)$, if $f(x) = \sqrt{x}$ and

$$g(x) = x^2 - 1.$$

Solution

Domain of $f = [0, \infty)$

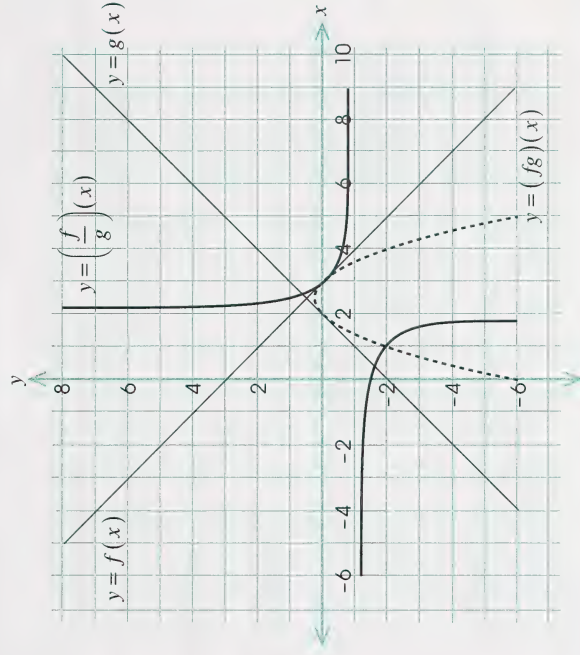
Domain of $g = (-\infty, \infty)$

Also, $g(x) = 0$ when $x = \pm 1$. Therefore, the domain of $y = \left(\frac{f}{g}\right)(x)$ is $\{x | x \geq 0 \text{ and } x \neq 1\}$. Generally, the graphs of the products and quotients of functions are beyond the scope of this section, but you will be reintroduced to them in Module 5. The following example will give you an idea of what to expect.

Example 4

If $f(x) = 3 - x$ and $g(x) = x - 2$, graph $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution



Notice that $\left(\frac{f}{g}\right)(x)$ is undefined for $x = 2$.

Part A



If you have access to a computer and the software Zap-a-Graph™, do question 4.

4. Explore products and quotients involving linear and quadratic functions.

Part B



Use a graphing calculator for question 5.

5. Graph $f(x) = x + 1$, $g(x) = x^2 - 3x + 2$, and their product and quotient. For what values of x is $\frac{f}{g}$ undefined?



Check your answers by turning to the Appendix.

Combinations of functions may be used to model real-world situations.

Example 5

Air temperature usually decreases uniformly with altitude h (in metres) until the tropopause is reached. The drop in temperature is approximately 0.6°C per 300 m. Assume the temperature at sea level is 20°C .

Write a function that describes the temperature at sea level as a function of altitude. Determine a function that describes the change in air temperature with altitude below the tropopause (10 000 m). Combine the previous two functions to obtain a function that could be used to predict the air temperature at any altitude below the tropopause. Then, graph the functions.

Solution

Let $f(h)$ be the temperature at sea level in degrees Celsius. Let $g(h)$ be the change in temperature in degree Celsius. Let $F(h)$ be the air temperature in degree Celsius. Let h be the altitude in metres.

Since the sea level temperature is independent of altitude, then $f(h) = 20$.

Since the drop in temperature is uniform, it can be represented in the form of $g(h) = kh$.

Since $g(300) = -0.6$, $-0.6 = 300k$

$$k = -0.002$$

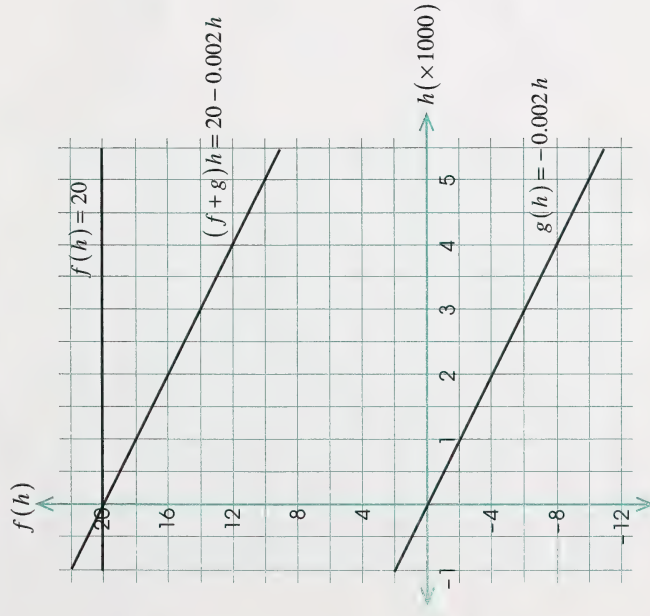
Thus, $g(h) = -0.002h$, where $h \leq 10\,000$.

The air temperature at a particular altitude h is the temperature at sea level plus the temperature change for that altitude.

$$\therefore F(t) = f(h) + g(h)$$

$$(f + g)(h) = 20 - 0.002h, \quad h \leq 10\,000$$

The graphs of functions are as follows:



6. Using the information in Example 5, at what altitude is the temperature 11°C ?



Check your answers by turning to the Appendix.

Functions are fundamental concepts to master in the language of mathematics.

Activity 2: Composition

In retail sales, the price the consumer ends up paying for an article depends on a number of factors: the cost of the article to the retailer, the retailer's costs of doing business, the profit expectations, and the taxes imposed by governments. Each of these considerations can be represented mathematically by a function; and these functions can be combined to set the final price of the article.

In this activity you will investigate a method of combining functions.

Example 1

A merchant decides the markup on garden tools will be 40% of the merchant's cost. The merchant knows that the Goods and Services Tax adds another 7% to the cost of these items to the consumer.

How can the merchant combine these two increases so that the after-tax price to the consumer can be calculated directly from the original cost? What would the after-tax cost be if the original cost to the merchant was \$10.00?

Solution

Let the original cost be x and the pre-tax price be $g(x)$, since the list price $g(x)$ is a function of the original cost x .

$$g(x) = x + 40\% \text{ of } x$$

$$g(x) = 1.40x$$

Now, the after-tax cost to the consumer C is a function of the list price $g(x)$.

$$\begin{aligned} C &= f(g(x)) \\ &= g(x) + 7\% \text{ of } g(x) \\ &= 1.07g(x) \\ &= 1.07(1.40x) \\ &= 1.498x \end{aligned}$$

From the previous relationship, $g(x) = 1.40x$.

This is the single function that can be applied in place of the other two. If the original cost was \$10.00, then

$$\begin{aligned} C &= f(g(10)) \\ &= 1.498(10) \\ &= 14.98 \end{aligned}$$

The cost is \$14.98.

Do you get the same answer by first calculating the list price and then the GST? Try it!



The process outlined in Example 1 of successively applying two functions to form a single function is called **composition**.

If $y = g(x)$ and $z = f(y)$, then $z = f(y) = f(g(x))$ is called the composition function of f and g . This composition function may be written as $f \circ g$; that is $(f \circ g)(x) = f(g(x))$.

Example 2

If $f(x) = x^2 + x$ and $g(x) = x - 1$, then find $f(g(-2))$, $g(f(-2))$, $f(g(x))$, and $g(f(x))$.

Solution

To find $f(g(-2))$, work from the inside out. Find $g(-2)$ first.

$$\begin{aligned} g(-2) &= -2 - 1 & \therefore f(g(-2)) &= f(-3) \\ &= -3 & &= (-3)^2 + (-3) \\ & & &= 6 \end{aligned}$$

To find $g(f(-2))$, evaluate $f(-2)$ first.

$$\begin{aligned} f(-2) &= (-2)^2 + (-2) & \therefore g(f(-2)) &= g(2) \\ &= 2 & &= 2 - 1 \\ & & &= 1 \end{aligned}$$

$$\begin{aligned}
 f(g(x)) &= (g(x))^2 + g(x) \\
 &= (x-1)^2 + (x-1) \\
 &= x^2 - 2x + 1 + x - 1 \\
 &= x^2 - x
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= f(x) - 1 \\
 &= x^2 + x - 1
 \end{aligned}$$

In Example 2, notice that the order in which the two functions are applied is critical; normally, $f(g(x)) \neq g(f(x))$. Also, notice that you can find the answers to the first two solutions by using the last two solutions.

$$f(g(x)) = x^2 - x$$

$$\begin{aligned}
 f(g(-2)) &= (-2)^2 - (-2) \\
 &= 6
 \end{aligned}$$

Use $g(f(x)) = x^2 + x - 1$ to find $g(f(-2))$. Is the answer 1?

You are now ready to try a few questions.

1. If $f(x) = x^2$, $g(x) = \frac{1}{x-1}$, and $h(x) = 2x - 4$, find the following:

- $f(g(0))$
- $f(h(x))$
- $h(f(g(3)))$
- $h(f(g(x)))$

2. If $f(x) = x + 2$ and $g(x) = x^2$, find x for which $f(g(x)) = g(f(x))$.

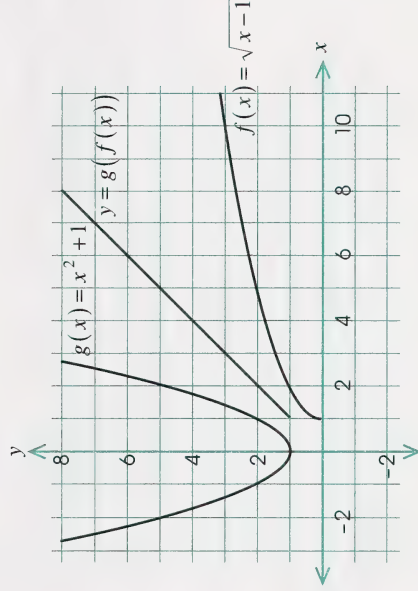


Check your answers by turning to the Appendix.

Example 3

If $f(x) = \sqrt{x-1}$ and $g(x) = x^2 + 1$, graph $y = f(x)$, $y = g(x)$ and $y = g(f(x))$. State the domain of all three functions.

Solution



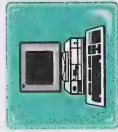
The domain of $f(x) = \sqrt{x-1}$ is $x \geq 1$, and its range is $y \geq 0$.



Even though $g(x) = x^2 + 1$ is defined for all real values of x , the composition function $g(f(x))$ is only defined when $f(x)$ is defined. As a consequence, the domain of $g(f(x))$ is also $x \geq 1$. What is the composition of g with f ?

$$\begin{aligned} g(f(x)) &= g(\sqrt{x-1}) \\ &= (\sqrt{x-1})^2 + 1 \\ &= x - 1 + 1 \\ &= x, \text{ where } x \geq 1 \end{aligned}$$

Graphing software packages will often not take the previous restrictions into account. If you have access to graphing software, enter the functions from Example 3 into your computer. If you did not get the same graph, why not?



Example 4

Given $f(x) = \sqrt{1-x}$ and $g(x) = \sqrt{3-x}$, determine the domain of $f(g(x))$.

Solution

$$\begin{aligned} f(g(x)) &= f(\sqrt{3-x}) \\ &= \sqrt{1-\sqrt{3-x}} \end{aligned}$$

Now $g(x)$ is defined for $x \leq 3$.

Also, $f(g(x))$ would only be defined if $\sqrt{3-x} \leq 1$ which suggests $3-x \leq 1$ or $x \geq 2$. If you combine these restrictions, the domain of $y = f(g(x))$ is $[2, 3]$.

- Determine the domain of $y = g(f(x))$ in Example 4.



Check your answers by turning to the Appendix.

Composition of functions is common in the physical sciences.

Example 5

The kinetic energy of an object of mass m is a function of its velocity v .

$$E(v) = \frac{mv^2}{2}$$

Also, suppose that the object had been dropped from rest. Its velocity v can be expressed as a function of time t .

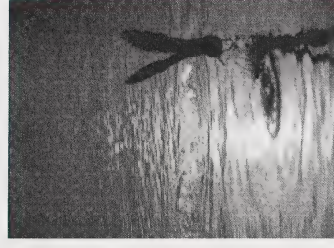
$$v(t) = gt$$

Express the kinetic energy of the object as a function of time.

Solution

$$E(v) = \frac{mv^2}{2}$$

$$\begin{aligned} E(v(t)) &= \frac{m(gt)^2}{2} \\ &= \frac{mg^2t^2}{2} \end{aligned}$$



4. A seagull dives into a lake and creates a circular wave which travels outward at a speed of 2 m/s. Determine the area enclosed by the wave as a function of time.



Check your answers by turning to the Appendix.

As consumer goods move from manufacturer to wholesaler to retailer, and finally, to consumer, costs are added to those articles at each stage. The final price is a composite of those cost increases. In mathematics, related functions that are applied consecutively can be combined into a single composite function.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Two concepts in the activities that you have just completed that are often confused are the composition of two functions and the product of two functions.

Example

If $f(x) = (x-1)^2$ and $g(x) = 2x+1$, determine $(fg)(x)$ and $(f \circ g)(x)$. Graph all four functions.

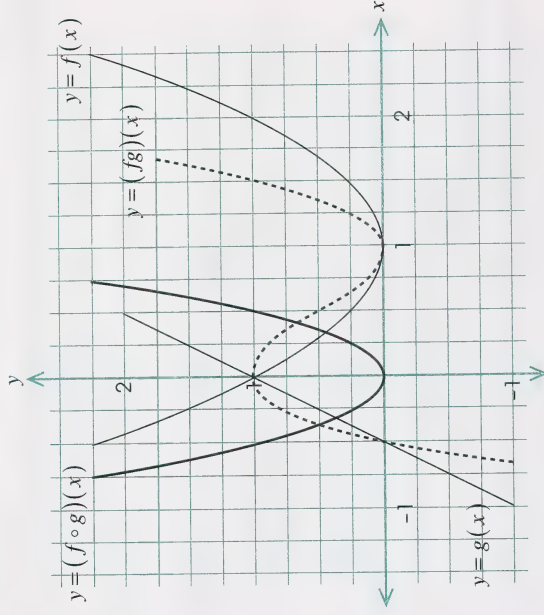
Solution

$(fg)(x)$ represents the product of the two functions.

$$\begin{aligned}(fg)(x) &= (x-1)^2(2x+1) \\ &= 2x^3 - 3x^2 + 1\end{aligned}$$

$(f \circ g)(x)$ represents the composition of f with g .

$$\begin{aligned}(f \circ g)(x) &= (g(x)-1)^2 \\ &= (2x+1-1)^2 \\ &= 4x^2\end{aligned}$$



Do the following questions.

1. If $f(x) = x-1$ and $g(x) = 2x-1$, determine $(fg)(x)$ and $(f \circ g)(x)$.
2. Using the functions defined in question 1, what values of x make $(fg)(x) = (f \circ g)(x)$?



Check your answers by turning to the Appendix.

Enrichment

In calculus, complicated-looking functions may often be simplified by expressing them as combinations of simpler functions. Consider the following examples.

Example 1

Express $y = (3x - 2)^2$ as the composition of two functions.

Solution

If $f(g(x)) = (3x - 2)^2$, then $g(x) = 3x - 2$ and $f(x) = x^2$.

Example 2

Express $y = \frac{x^3}{x^3 + 1}$ as a composite function of the form $f \circ g$.

Solution

Let $g(x) = x^3$ and $f(x) = \frac{x}{x+1}$. Therefore, $(f \circ g)(x) = \frac{x^3}{x^3 + 1}$.

Express the following functions as a composition of two or more functions.

1. $y = (x + 2)^4$

2. $y = \sqrt{x - 7}$

3. $y = \frac{3}{x + 4}$

4. $y = (\sqrt{x + 3})^5$



Check your answers by turning to the Appendix.

Conclusion

In this section you have been shown how to combine functions algebraically (using the arithmetic operations), and how to combine functions using a procedure called composition. In both instances you have seen that the domains and the graphs of these combinations are dependent on the original functions.

You also discovered that complicated functions may be represented through the combinations of simpler functions (functions that have been studied in previous courses).

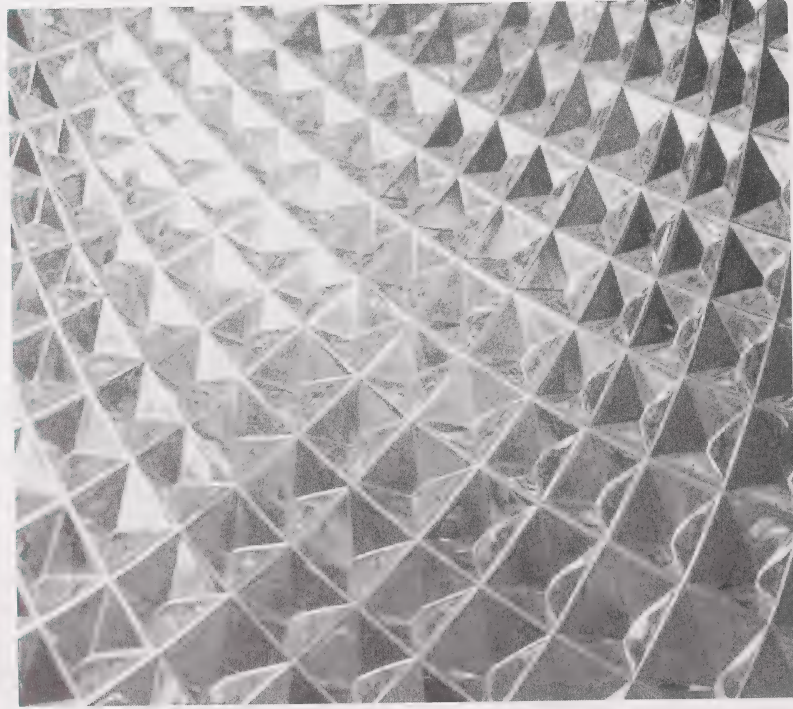
The shapes and patterns of rocks along a stream are a function of countless natural forces. Mathematics attempts to explain the regularities and patterns in nature. Simple functions, in which each may represent a single process, are combined to explore a more complex process in nature.

Assignment



You are now ready to complete the section assignment.

Section 3: Transformations and Symmetries



Often in architecture, the same geometric forms recur. Patterns are achieved by repeating those forms horizontally or vertically. The same geometric forms and their mirror images may occur in different sizes or in different positions in the same structure. Visually pleasing results may also be attained through symmetry.

In mathematics, your ability to analyse functions and their graphs will depend on your knowledge of similar transformations: how to shift graphs horizontally or vertically, how to reflect graphs in the x - and y -axes or across lines such as $y = x$, and how to stretch or compress graphs. Also, you will be expected to recognize, from the equations of functions, which functions are symmetric with respect to the x -axis, the y -axis, or the origin. Many of the functions you will be dealing with will be variations of those you have studied in previous courses.

In this section, you will study how to alter the equations of functions in order to translate graphs horizontally and vertically, and you will determine what changes are required in the equations to reflect those graphs across the x - and y -axes or $y = x$. You will also vertically dilate or compress a curve. All of these effects will be related to the general function $y = af[b(x \pm c)] \pm d$ and the parameters a , b , c , and d .

You will be expected to test for symmetry and sketch curves from that analysis.

Activity 1: Translations

Have you been at an airshow and seen jets flying in tight formation? Often, the planes fly so close together, and manoeuvre with such precision that they appear to be a single aircraft. The effect appears like a pattern formed by stamping out an image in the sky by moving a single plane to the left and right or forwards and back.



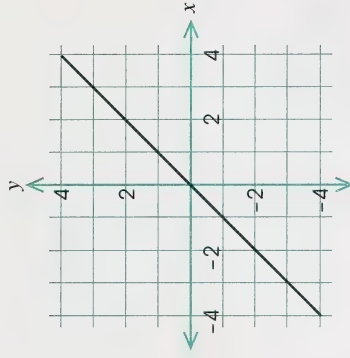
In mathematics, how would you similarly reposition the graphs of functions without changing their shape or size? What changes would you have to make in the equations of those functions?



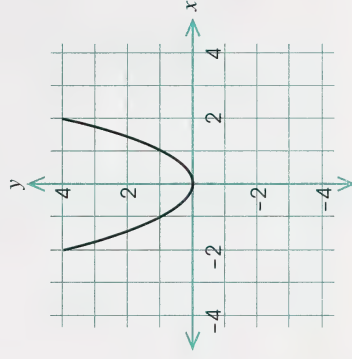
A **translation** is a horizontal or vertical shift of the graph of a function.

In previous mathematics courses you studied how to sketch common functions like $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, $y = \frac{1}{x}$, and $y = 2^x$.

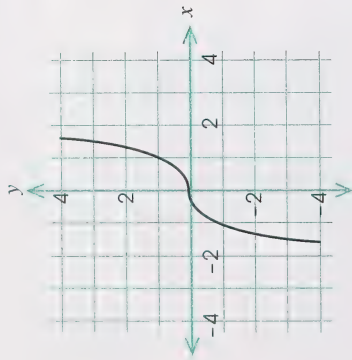
- $y = x$



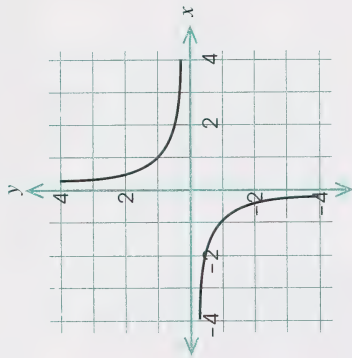
- $y = x^2$



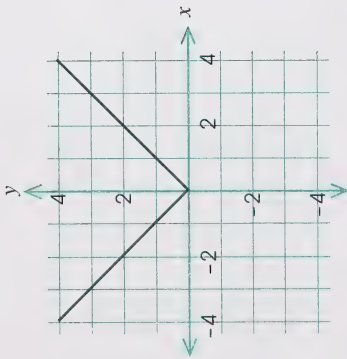
- $y = x^3$



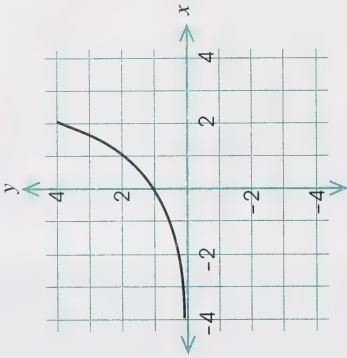
- $y = \frac{1}{x}$



- $y = |x|$



- $y = 2^x$



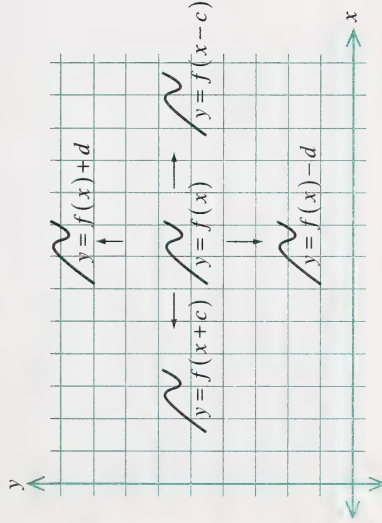
You were also shown how to translate these and other functions, both horizontally and vertically, by altering their equations.

Recall that for the function $y = f(x)$, if parameters c and d are positive, then the following is true.

- $y = f(x - c)$ moves the graph of $y = f(x)$ to the right c units.
- $y = f(x + c)$ moves the graph of $y = f(x)$ to the left c units.
- $y = f(x) + d$ moves the graph of $y = f(x)$ upward d units.
- $y = f(x) - d$ moves the graph of $y = f(x)$ downward d units.



The following graph shows the effect of parameters c and d .



Example 1



Using a graphing calculator, sketch each of the

functions $f(x) = \sqrt{4 - x^2}$,

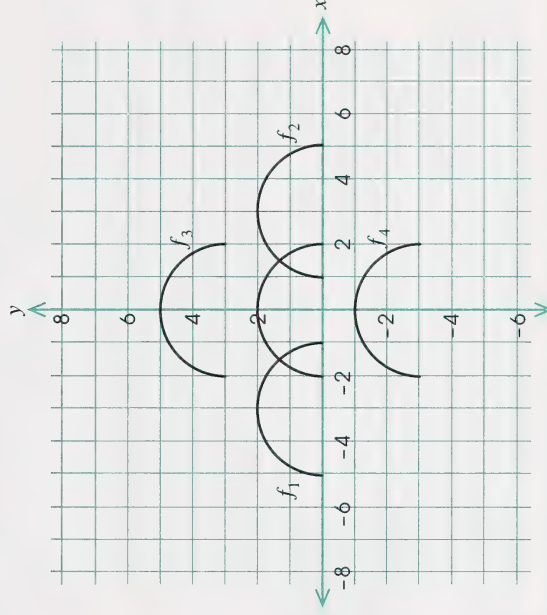
$$f_1(x) = f(x + 3) = \sqrt{4 - (x + 3)^2},$$

$$f_2(x) = f(x - 3) = \sqrt{4 - (x - 3)^2},$$

$$f_3(x) = f(x) + 3 = \sqrt{4 - x^2} + 3, \text{ and}$$

$f_4(x) = f(x) - 3 = \sqrt{4 - x^2} - 3$. Describe how these graphs are related.

Solution



The graph of $y = f_1(x)$ is the graph of $y = f(x)$ translated 3 units to the left.

The graph of $y = f_2(x)$ is the graph of $y = f(x)$ translated 3 units to the right.

The graph of $y = f_3(x)$ is the graph of $y = f(x)$ translated 3 units upward.

The graph of $y = f_4(x)$ is the graph of $y = f(x)$ translated 3 units downward.

You may have observed that the same translations could have been obtained by combining functions. For instance, if $f(x) = \sqrt{4-x^2}$ and $g(x) = x+3$ and $h(x) = 3$, then $f_1(x) = f(g(x))$ and $f_3(x) = f(x) + h(x)$.

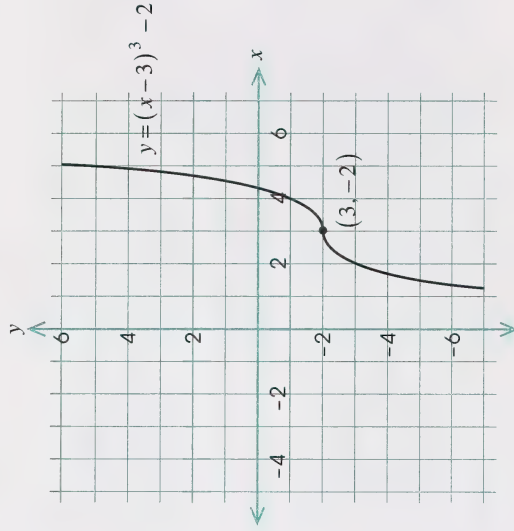
1. How can $f_2(x)$ and $f_4(x)$ from Example 1 be obtained by combining functions?



Check your answers by turning to the Appendix.

Example 2

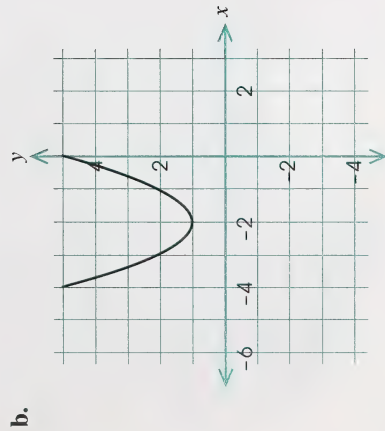
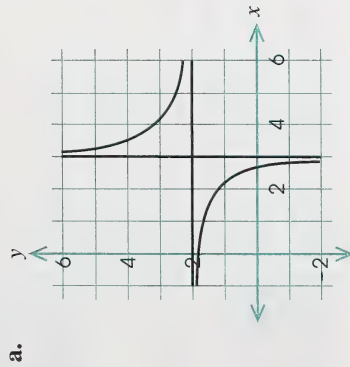
If the graph is congruent to that of $y = x^3$, what is the function graphed?



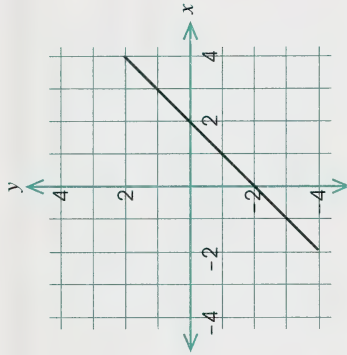
Solution

Since $(3, -2)$ corresponds to $(0, 0)$ on $f(x) = x^3$, the graph of $y = x^3$ must be translated twice to obtain the second curve: 3 units to the right and 2 units downward. Therefore, the new function must be $y = f(x-3) - 2$ or $y = (x-3)^3 - 2$.

2. Assuming that the graphs given are congruent to one of $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, or $y = \frac{1}{x}$, write the equations of these functions and describe the translations each graph has undergone.



c.



3. Sketch each function by using horizontal and vertical translations.

a. $y = |x + 1| - 2$ b. $y = \frac{1}{x - 1} + 3$ c. $y = \sqrt{x - 3} + 4$

4. The height h (in metres) of a ball thrown upward from ground level, after t seconds, is given by $h = -5t^2 + 4t$. What would the equation be if it were thrown upward at the same speed but from the bottom of a well 4 m below ground level?



Check your answers by turning to the Appendix.

Aircraft flying in formation keep their position relative to each other. Their positions are determined by the pattern they wish to display. A function can be fixed in the coordinate plane or translated to a desired position by moving it horizontally or vertically.

Activity 2: Reflections

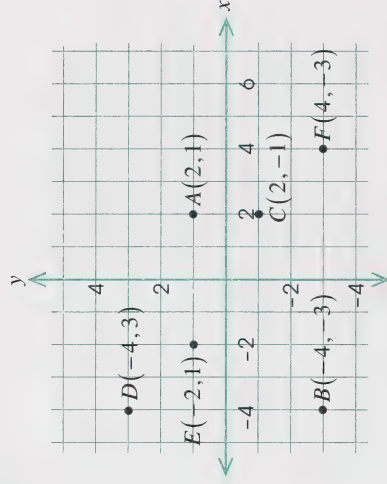


Besides translations, there are additional **transformations** of functions you will need to investigate to prepare for calculus. A transformation is a change or alteration in the form of a function. In this activity you will discover how the equation of a function must be altered to **reflect** its graph in either the x -axis or the y -axis, and also how to reflect it in the line $y = x$.

Example 1

Graph the points $A(2, 1)$, $B(4, -3)$, $C(2, -1)$, $D(-4, 3)$, $E(-2, 1)$, and $F(4, -3)$. Which pairs of points are mirror images in the x -axis? Which pairs of points are mirror images in the y -axis? Why?

Solution



Points A and C are mirror images of each other in the x -axis, and points B and D are mirror images of each other. Notice that the x -coordinates of these pairs of points are the same, but the y -coordinates are opposites. Also, if you joined points, like A and C (which are reflections in the x -axis), the x -axis would bisect that segment at right angles. This is exactly what happens when you stand in front of a mirror; the line segment joining you and your image is bisected at right angles by the plane of the mirror.

Points A and E, and points B and F, are mirror images in the y-axis. Notice this time that the x-coordinates of these pairs of points are opposites and the y-coordinates are the same.



In the next example you will see that because (x, y) and $(-x, y)$ are **mirror images in the y-axis**, the graph of the function $y = f(x)$ reflects into the graph of $y = f(-x)$ across the y-axis.

Similarly, because (x, y) and $(x, -y)$ are **mirror images in the x-axis**, the graph of the function $y = f(x)$ reflects into the graph of $y = -f(x)$ across the x-axis.

Example 2

Determine the equations of the functions that are reflections of $y = x^2 - 4x$ in the y-axis and in the x-axis. Sketch each function.

Solution

Because $y = f(x)$ reflects into $y = f(-x)$ across the y-axis, the image of $y = x^2 - 4x$ is as follows:

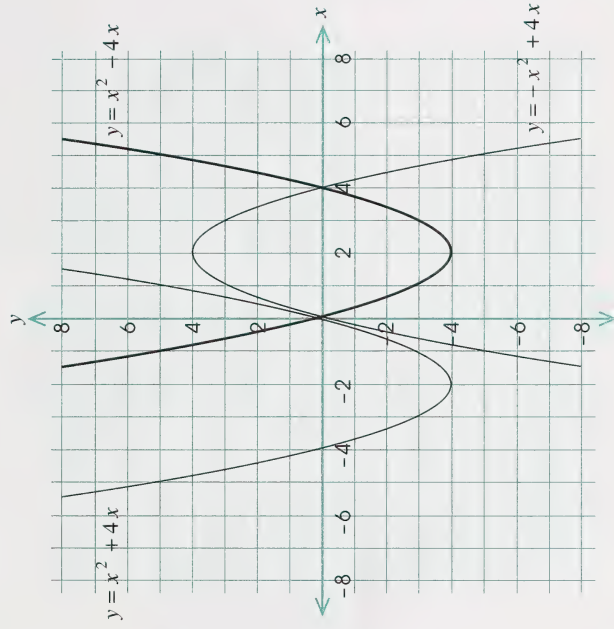
$$\begin{aligned} y &= (-x)^2 - 4(-x) \\ &= x^2 + 4x \end{aligned}$$

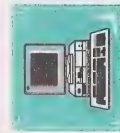
Replace x with $-x$.

Because $y = f(x)$ reflects into $y = -f(x)$ across the x-axis, the image of $y = x^2 - 4x$ is as follows:

$$\begin{aligned} -y &= x^2 - 4x \\ y &= -x^2 + 4x \end{aligned}$$

Replace y with $-y$.





1. Determine the equations of the reflections of each function in the x -axis and in the y -axis. Check your graphs by using a graphing calculator or the graphing software Zap-a-Graph™.

- a. $y = 2^x$ b. $y = x$
 c. $y = |x|$ d. $y = x^3$
 e. $y = -x^2 - 5x + 1$ f. $y = x^3 - x$

Check your answers by turning to the Appendix.

How are equations of functions altered by both a translation and a reflection?

Example 3

Graph and compare $y = x^3$ and $y = -(x-3)^3 + 2$.

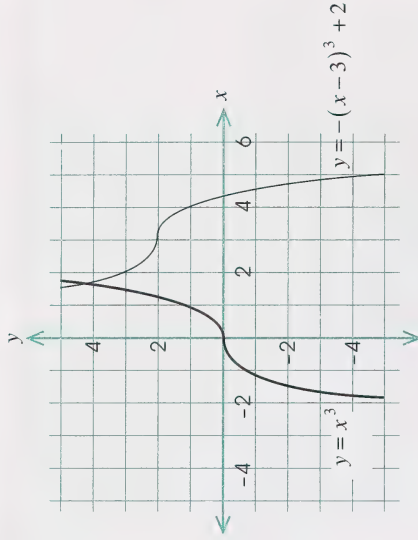
Solution

$y = (x-3)^3$ is $y = x^3$ translated 3 units to the right.

$y = -(x-3)^3$ is $y = (x-3)^3$ reflected in the x -axis.

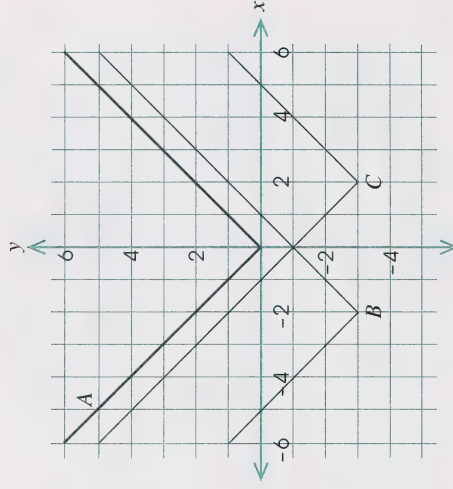
$y = -(x-3)^3 + 2$ is $y = -(x-3)^3$ translated 2 units upward.

Therefore, the graph of $y = x^3$ has been translated 3 units to the right, reflected in the x -axis, and translated 2 units upward.



Example 4

Determine the equation of graphs B and C assuming it is a transformation of graph A ($y = |x|$).



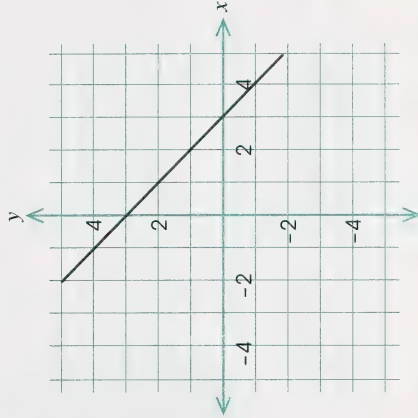
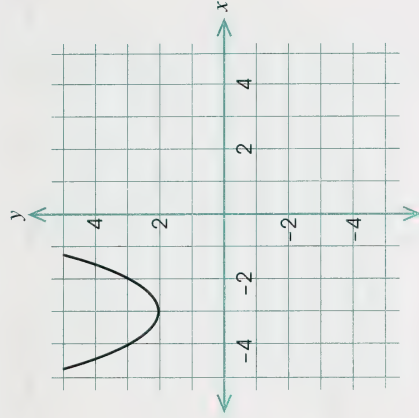
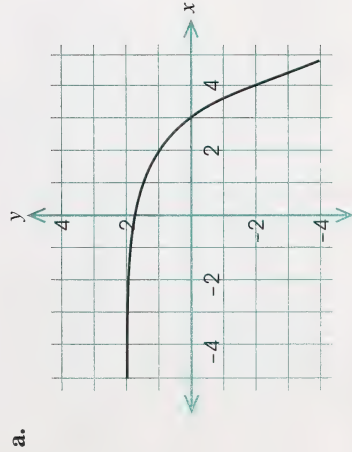
Solution

To obtain graph *B*, translate $y = |x|$ to the left 2 units and down 3 units. Therefore, $y = |x + 2| - 3$.

To obtain graph *C*, reflect $y = |x + 2| - 3$ in the y -axis; that is, replace x by $-x$. Therefore, $y = |-x + 2| - 3$.

Notice that graph *C* could have been obtained directly by translating $y = |x|$ to the right 2 units and down 3 units. The equation for graph *C* could be $y = |x - 2| - 3$. These results are equivalent?

- Determine the equations of the following functions assuming that their graphs are congruent to one of $y = x$, $y = x^2$, $y = x^3$, $y = 2^x$, $y = |x|$, or $y = \frac{1}{x}$. Check your answers using a graphing calculator or graphing software.



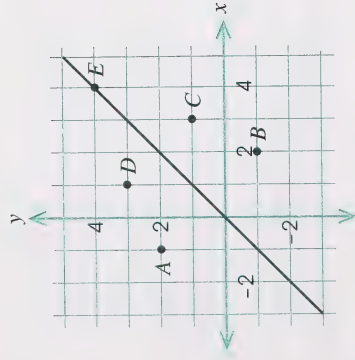
Check your answers by turning to the Appendix.

In Mathematics 20, you studied the inverse of a function and the relationship between the graphs of a function and its inverse.

Example 5

Plot $A(-1, 2)$, $B(2, -1)$, $C(3, 1)$, $D(1, 3)$, and $E(4, 4)$ on a graph. How are these points related with respect to the line $y = x$?

Solution



A and B , and C and D , are mirror images of each other in the line $y = x$.



Point E is its own image. In each case the **coordinates of the points were interchanged** to form their images.

The fact that $y = x$ acts as a mirror can be proven for A and B as follows. Join points A and B . The midpoint of AB is $(0.5, 0.5)$ which lies on $y = x$. Because the slope of AB is -1 , and the slope of $y = x$ is 1 , $y = x$ intersects AB at right angles. Thus, B is the reflection of A (and vice versa) since B is the same distance from the mirror as A is! The proof that this is always true is left to you as an exercise.

3. Prove that $y = x$ is the perpendicular bisector of the segment PQ with $P(a, b)$ and $Q(b, a)$; hence, P and Q are images of each other in $y = x$.



Check your answers by turning to the Appendix.

Example 6

Discuss the graphs of $f(x) = x^2 + 3$ and its image reflected in $y = x$.

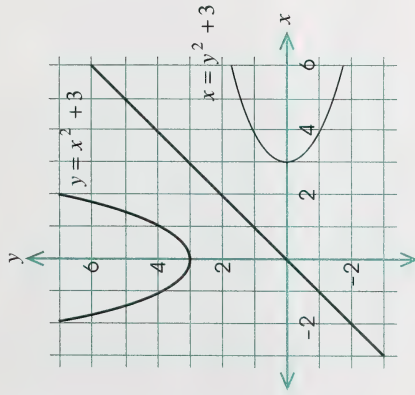
Solution

To reflect the graph of the function $y = x^2 + 3$ in $y = x$, as in Example 5, simply replace x with y and y with x .

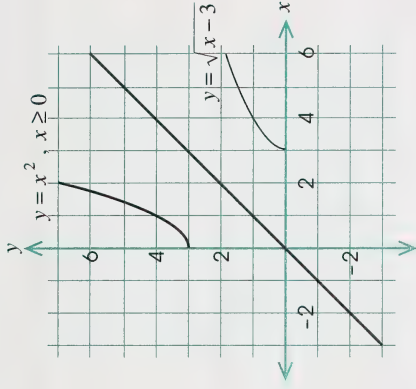
Original function: $y = x^2 + 3$

Image: $x = y^2 + 3$

Notice from the graph that the relation $x = y^2 + 3$ is not a function, since y is not uniquely determined for $x > 3$.



If you were to restrict the original function to $y = x^2 + 3$, $x \geq 0$, then its image would be $x = y^2 + 3$, $y \geq 0$. The negative portions of each graph would be eliminated. As a result, the image could be rewritten as $y = \sqrt{x - 3}$.



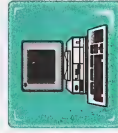
Written in functional notation, the image of

$$f(x) = x^2 + 3, x \geq 0 \text{ would be } f^{-1}(x) = \sqrt{x - 3}.$$

This notation is read **the inverse function of x** , and is restricted to those cases for which the inverse of a given function, obtained by interchanging x and y , is itself a function.

You may recall that the inverse will undo what the function does. For example, the function takes 4 into 19; that is, $f(4) = 19$. The inverse will take 19 back into 4: $f^{-1}(19) = 4$.

Part A



If you have access to Zap-A-Graph™, explore question 4 using the Transform features of the program.

4. Determine the equation of the image of each function in $y = x$.
If the image is a function, write it using $f^{-1}(x)$.

- a. $f(x) = x^2$ b. $f(x) = x^2 + 3x$
c. $f(x) = x^3$ d. $f(x) = |x|$

Part B



Use a graphing calculator to confirm your results to question 4.



Check your answers by turning to the Appendix.

In the photograph, the bridge and its reflection in the water are symmetric about a horizontal line, just like a function and its reflection about the x -axis.



Activity 3: Expansion and Compression

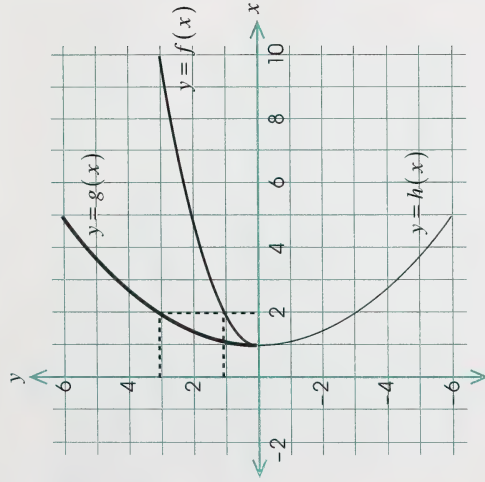


Have you ever watched a television set that stretched or expanded the picture vertically or perhaps distorted the picture by squeezing or compressing it? In this activity, you will discover the changes necessary in the equation of a function to expand or compress its graph.

Example 1

Draw and compare the graphs of $f(x) = \sqrt{x-1}$, $g(x) = 3\sqrt{x-1}$, and $h(x) = -3\sqrt{x-1}$.

Solution



The graph of $y = g(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of 3. Because $g(x) = 3f(x)$, for each value of x in the domain, the y -coordinate of $y = g(x)$ will be three times larger than the y -coordinate of $y = f(x)$. For instance, when $x = 1$, $f(1) = 1$ and $g(1) = 3$; or if $x = 5$, $f(5) = 2$, and $g(5) = 6$. The graph of $y = h(x)$ is the reflection of $y = g(x)$ in the x -axis.

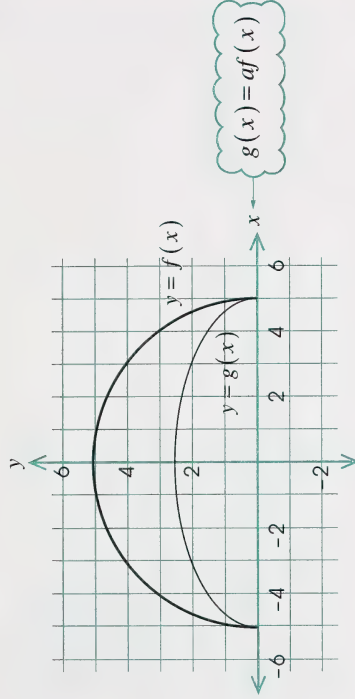


The graph of $y = af(x)$ is the graph of $y = f(x)$ stretched vertically $|a|$ times.

Example 2

Compare $f(x) = \sqrt{25 - x^2}$ with $g(x) = 0.5\sqrt{25 - x^2}$.

Solution



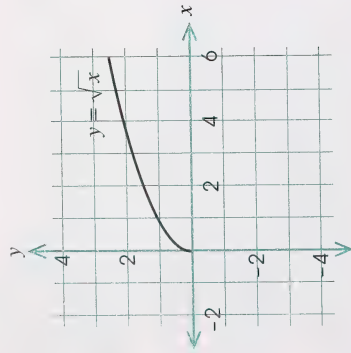
Because $|a| = 0.5$, the graph of $y = \sqrt{25 - x^2}$ has been compressed to one-half its vertical size to give $y = 0.5\sqrt{25 - x^2}$.

Example 3

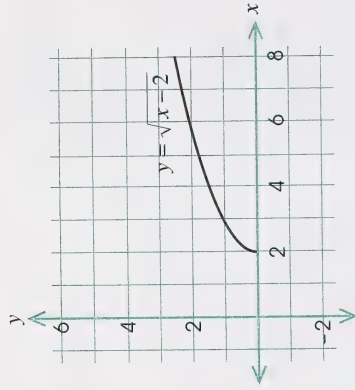
Show, by a series of transformations, how the graph of $y = -3\sqrt{x - 2} + 4$ can be obtained from $y = \sqrt{x}$.

Solution

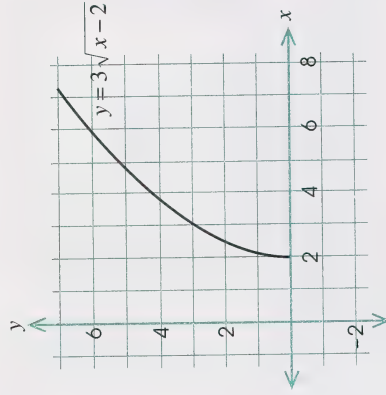
Graph A: Draw the original graph.



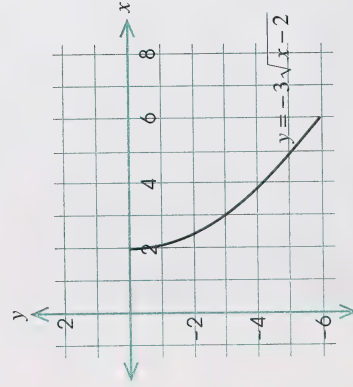
Graph B: Translate 2 units to the right.



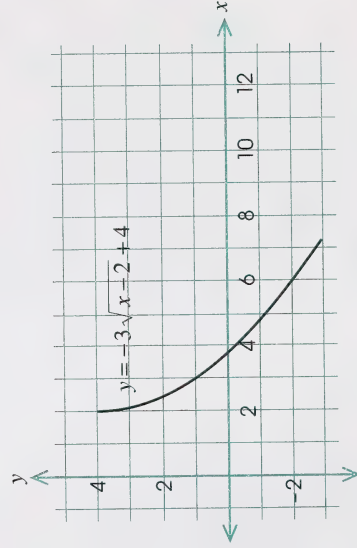
Graph C: Stretch to three times the original graph.



Graph D: Reflect in x-axis.



Graph E: Vertical translation of 4 units.

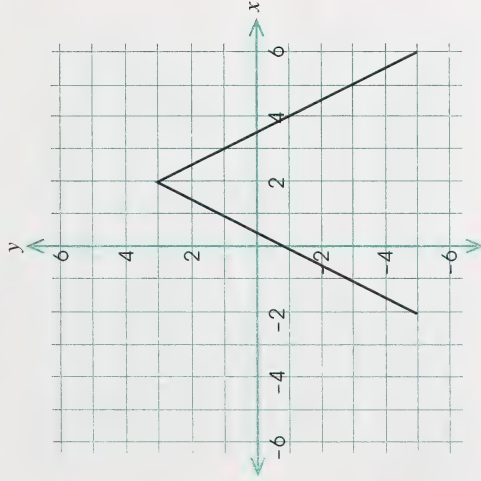


1. What is the effect of increasing a on the appearance of function such as $y = x^2$ or $y = |x|$?



Use a graphing calculator to check your answers to questions 2 and 3.

2. What is the equation of the function graphed?



3. Sketch each of the following functions by considering the transformations of $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, $y = \sqrt{x}$, or $y = \frac{1}{x}$. Do not use a table of values.

a. $y = \frac{3}{x-2}$ b. $y = 5x + 1$ c. $y = -3(x-2)^3 + 1$



Check your answers by turning to the Appendix.

In the early days of television, the picture appeared distorted when the vacuum tubes started to fail in a television set. The picture was often stretched or compressed. In this activity you explored how the equation of a function can be changed to stretch or compress its graph.

Activity 4: Symmetry

Symmetry is common in nature. The human body, for example, is symmetric about a vertical plane; certain plants or flowers can be split in two, with one side mirroring the other.



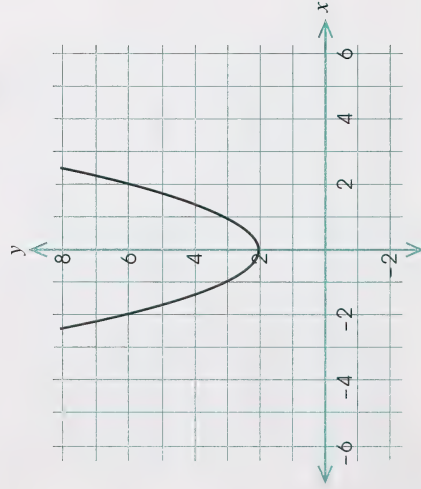
In mathematics, it is to your advantage to recognize symmetries when they occur. To graph certain functions and relations, it may be necessary to sketch only a portion of the curve, the remainder being reflections of the portions already known.

Example 1

Reflect $y = x^2 + 2$ in the y -axis.

Solution

To reflect a curve in the y -axis, replace x with $-x$. As you see, in this case, $y = (-x)^2 + 2$ is the same as $y = x^2 + 2$. This suggests that $y = x^2 + 2$ is its own mirror image in the y -axis. Therefore, $y = x^2 + 2$ is symmetric with respect to the y -axis!



A graph is **symmetric with respect to the y -axis** if its equation is unchanged when x is replaced by $-x$.

- Which of the following are symmetric with respect to the y -axis?

- $y = |x|$
- $y^2 = -x^3$
- $y = -x^4 - 5x^2$
- $y = \frac{1}{x}$

- Each of the following functions has a vertical axis of symmetry. Use your knowledge of transformations to find the axis of symmetry in each case.

- $y = 2(x - 4)^2$
- $y = -|x + 1|$
- $y = 2x^2 - 4x$



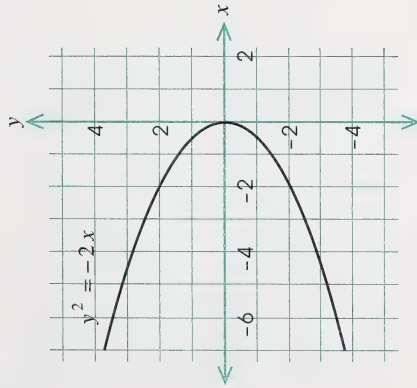
Check your answers by turning to the Appendix.

Example 2

Reflect $y^2 = -2x$ in the x -axis.

Solution

To reflect a curve in the x -axis, replace y by $-y$. In this example, $y^2 = -2x$ is the same as $(-y)^2 = -2x$. Therefore, $y^2 = -2x$ is its own image in the x -axis. This curve is symmetric with respect to the x -axis.



A curve is **symmetric with respect to the x -axis** if the equation of that curve is unchanged when y is replaced by $-y$.

3. Which of the following are symmetric with respect to the x -axis?

- $y^2 = x - 2$
- $x = |y|$
- $y = x^4 - 1$
- $y^2 = x^2$

4. Each of the following relations has a horizontal axis of symmetry. Use your knowledge of transformations to find the axis of symmetry in each case.

- a. $x = 2(y - 4)^2$ b. $x = y^2 + 4$ c. $x = |y - 2|$

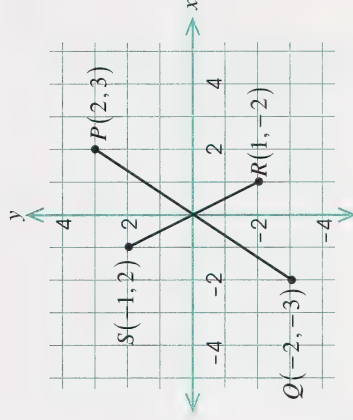


Check your answers by turning to the Appendix.

Example 3

Graph $P(2, 3)$, $Q(-2, -3)$, $R(1, -2)$, and $S(-1, 2)$. Which pairs of points are images across the origin? Why?

Solution



P and Q are images of each other, reflected across the origin. If P and Q are joined, as in the diagram, the origin is the midpoint of that segment. Similarly, R and S are images of each other, reflected across the origin.



The graph of a function or relation is said to be **symmetric with respect to the origin**, if the equation of that function or relation remains unchanged when (x, y) is replaced by $(-x, -y)$.

Example 4

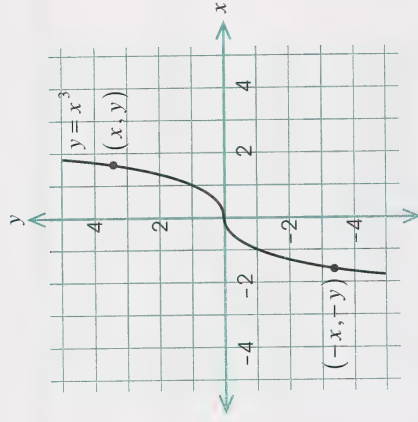
Sketch $y = x^3$. Describe its symmetries.

Solution

Replace (x, y) by $(-x, -y)$ in the equation $y = x^3$.

$$\begin{aligned}(-y) &= (-x)^3 \\ -y &= -x^3 \\ y &= x^3\end{aligned}$$

Because the resulting equation is the same as the original equation, the graph must be symmetric with respect to the origin; each point (x, y) is reflected into $(-x, -y)$.



5. Which of the following curves are symmetric with respect to the origin?

- $y = x$
- $y = x^5$
- $y^3 = x$
- $y^3 = 2x^5 - x$

6. Which of the following curves are symmetric with respect to the x -axis? the y -axis? the origin? Draw the graph of each.

- a. $y^2 = x^2$
- b. $y = \frac{1}{x}$
- c. $y^2 = \frac{1}{x}$
- d. $x^2 + y^2 = 25$
- e. $x^2 - y^2 = 1$



Check your answers by turning to the Appendix.

Symmetry enhances the beauty of a flower. There is a certain beauty in the graphs of functions which display symmetries. You may wish to review how to determine, from its equation, whether a graph is symmetric about the axes or the origin.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Often, it is difficult to visualize the graphs of relations written in the form $x = f(y)$. Is there an easy way of drawing relations such as

$$x = y^2, \quad x = y^3 - y^2, \quad \text{and} \quad x = |y|?$$

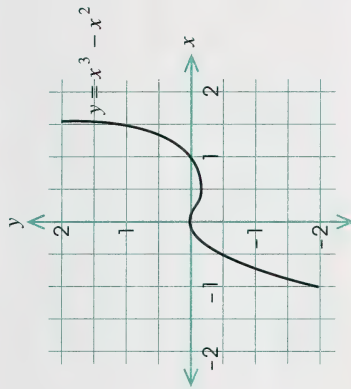
Example

Sketch $x = y^3 - y^2$.

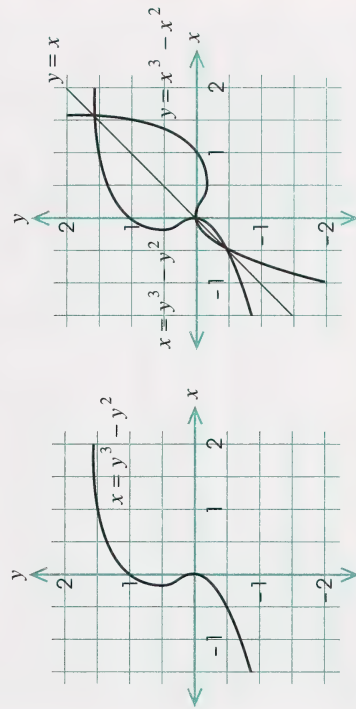
Solution

It is easier to think in terms of $y = f(x)$. Begin by sketching

$$y = x^3 - x^2.$$



Reflect this graph across $y = x$ to obtain $x = y^3 - y^2$. You might select points from $y = x^3 - x^2$ such as $(0, 0)$, $(1, 0)$, $(2, 4)$, and $(-1, -2)$. Plot their images $(0, 0)$, $(0, 1)$, $(4, 2)$, and $(-2, -1)$ to get started. Your end result and the relationship between the two graphs should be as follows:



Use this approach to sketch the following.

1. $x = y^2$

2. $x = y^3 + y^2$

3. $x = |y|$



Check your answers by turning to the Appendix.

Enrichment

Part A



If you have access to the computer graphing package Zap-a-Graph™, investigate the changes in the equation of a curve as you rotate it through 90° , 180° , and 270° . How is the graph obtained by a 270° rotation related to the reflection of the original curve in $y = x$?

Part B

Using graph paper, carefully draw $y = x^3$. By tracing this graph onto a second sheet, show the curve as it is rotated through 90° , 180° , and 270° . How is the graph obtained by a 270° rotation related to the reflection of the original curve in $y = x$?

Conclusion

In this section, you analysed and graphed functions or relations by taking advantage of the basic transformations: horizontal and vertical translations, reflections across the x - or y -axes and the line $y = x$, and dilations and compressions.

Often, complex functions can be drawn or viewed as much more basic functions, simply repositioned, reflected, or rescaled. Much of the time you will be able to assemble the complicated from the simple, or take the complicated apart into well-know elements.

Whenever you are to sketch a function or relation, ask yourself if a portion of the graph can be used to form the entire graph by reflecting that portion across either axis or the origin.

An architect strives to create both functional and visually appealing designs. Patterns often occur in those designs; patterns can be achieved by repeating the same geometric form, by changing its scale each time it occurs, or by reflection and symmetry.

An artist, a designer, and an architect are a few of the many professionals that must consider transformations and symmetries.

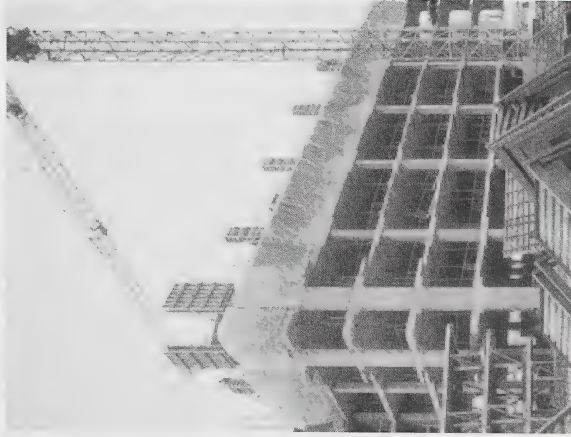
Assignment



You are now ready to complete the section assignment.

Module Summary

You have just completed the first module of Mathematics 31. In this module, you worked on (or reviewed) material that will be essential to your study of calculus. The success of your future work with functions and relations, and their graphs, will depend on your ability to pinpoint specific regions of the plane or portions of those graphs, to solve inequalities, and to express the solutions of those inequalities using set-builder notation and interval notation.



In addition, you were shown how to transform and combine functions into more complicated structures. The relationships between the constituent functions and the resulting combinations and transformations will enable you to analyse complex functions and their graphs with more ease.

Another essential skill in analysing functions is the recognition of the conditions under which symmetry occurs: in the x -axis, the y -axis, and the origin.

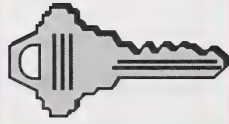
Just as a construction company would not consider building on anything but a firm foundation, you must master the topics in this module to lay a firm foundation of concepts and procedures to facilitate your study of calculus. Review those sections and activities with which you had difficulty.

Final Module Assignment



You are now ready to complete the final module assignment.

APPENDIX



Glossary

Suggested Answers

Glossary

Absolute value of a real number: the magnitude of x , written $|x|$;

$$|x| = x \text{ if } x \geq 0, \text{ and } |x| = -x \text{ if } x < 0$$

Closed interval: an interval which includes both of its endpoints, written $[x, y]$ if x and y are the endpoints

Composition of f with g : written $f \circ g$ or $y = f(g(x))$ is the function obtained by replacing x with $g(x)$ in $y = f(x)$

Intersection of two sets: consists of only those elements common to the two sets, written $A \cap B$ if A and B are the two sets

Interval of real numbers: the set of all real numbers between two given values, called the endpoints of the interval

One, both, or neither of the two endpoints is included in the set of numbers.

Open interval: an interval which includes neither of its endpoints, written (x, y) if x and y are the endpoints

Transformation: a change from one expression or figure to another

Translation: a horizontal or vertical shift of a figure

Union of two sets: consists of only those elements belonging to either of the two sets or both, written $A \cup B$ if A and B are the two sets

Suggested Answers

Section 1: Activity 1



$$\{x | -3 \leq x \leq 5\}$$



$$\{x | 2 < x < 3\}$$

3. $\{x | -3 \leq x < 0\}$



Section 1: Activity 2

1. $x = \pm 4$

2. $\{x | -4 < x < 4\}$



3. $|4 - x| < 7$

$-7 < 4 - x < 7$

$-11 < -x < 3$

$11 > x > -3$

The solution set is $\{x | -3 < x < 11\}$ or $(-3, 11)$.

4. **Case 1:** Assume $x + 5 > 0$; that is $x > -5$

$$\frac{x-3}{x+5}(x+5) \leq 3(x+5)$$

$$x-3 \leq 3x+15$$

$$-2x \leq 18$$

$$x \geq -9$$

Reconciling $x > -9$ with the assumption that $x > -5$, the interval $(-5, \infty)$ is part of the solution.

Case 2: Assume $x + 5 < 0$; that is $x < -5$

$$\frac{x-3}{x+5}(x+5) \geq 3(x+5)$$

$$x-3 \geq 3x+15$$

$$-2x \geq 18$$

$$x \leq -9$$

Reconciling $x \leq -9$ with the assumption that $x < -5$, the interval $(-\infty, -9]$ is part of the solution.

Therefore, the solution set is $(-5, \infty) \cup (-\infty, -9]$.

5. $\frac{|x-3|}{|x+2|} \leq 2$ ①

$|x-3| \leq 2|x+2|$ ②

When dealing with absolute value, it is not certain whether the expressions within the absolute value symbols are positive, negative, or zero.

Possible Combinations		
	$x - 3$	$x + 2$
Case 1	+	+
Case 2	+	-
Case 3	-	+
Case 4	-	-

Because $x-3$ could be zero, you will incorporate this possibility in Case 1 and Case 2.

Case 1: $x-3 \geq 0$ and $x+2 > 0$. For both to be non-negative, $x \geq 3$. You will search for solutions in the interval $[3, \infty)$.

According to the definition of absolute value, because both binomials are non-negative, $|x - 3| = x - 3$ and $|x + 2| = x + 2$.

Therefore, substitute in (2).

$$\begin{aligned}(x - 3) &\leq 2(x + 2) \\ x - 3 &\leq 2x + 4 \\ -x &\leq 7 \\ x &\geq -7\end{aligned}$$

Since you are searching for solutions in $[3, \infty)$, the interval obtained from this case is $[3, \infty)$.

Case 2: $x - 3 \geq 0$ and $x + 2 < 0$. This implies $x \geq 3$ and $x < -2$. This is impossible. Go to Case 3.

Case 3: $x - 3 < 0$ and $x + 2 > 0$. This implies $x < 3$ and $x > -2$. This time you will search for additional solutions in the interval $(-2, 3)$. As before, $|x + 2| = x + 2$, but $|x - 3| = -(x - 3)$ because $x - 3$ is negative.

Therefore, substitute in (2).

$$\begin{aligned}-(x - 3) &\leq 2(x + 2) \\ -x + 3 &\leq 2x + 4 \\ -3x &\leq 1 \\ x &\geq -\frac{1}{3}\end{aligned}$$

Since you are searching for solutions in $(-2, 3)$, the interval obtained from this case is $[-\frac{1}{3}, 3)$.

Case 4: $x - 3 < 0$ and $x + 2 < 0$. For both to be negative, then $x < -2$. This time you will search for additional solutions in the interval $(-\infty, -2)$.

Because both binomials are negative, $|x - 3| = -(x - 3)$ and $|x + 2| = -(x + 2)$.

Therefore, substitute in (2).

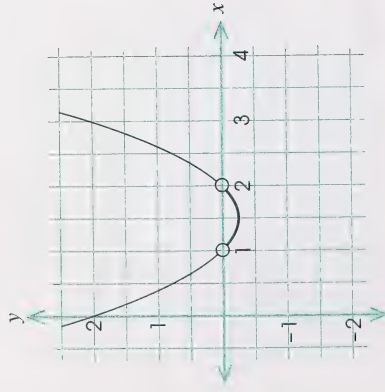
$$\begin{aligned}-(x - 3) &\leq -2(x + 2) \\ -x + 3 &\leq -2x - 4 \\ x &\leq -7\end{aligned}$$

Since you are searching for solutions in $(-\infty, -2)$, the interval $(-\infty, -7]$ is obtained.

Therefore, combining the solutions from Cases 1, 3, and 4, you get $(-\infty, -7] \cup [-\frac{1}{3}, 3) \cup [3, \infty)$ or $(-\infty, -7] \cup [-\frac{1}{3}, \infty)$.

Section 1: Activity 3

1. The portion of the graph below the x -axis occurs for values of x in the interval $(1, 2)$.



2.
$$2x^2 > 5x - 2$$
$$2x^2 - 5x + 2 > 0$$
$$(2x - 1)(x - 2) > 0$$

Case 1: Both factors are positive.

$$2x - 1 > 0 \quad \text{and} \quad x - 2 > 0$$

$$x > 0.5 \qquad x > 2$$

Therefore, $x > 2$.

Case 2: Both factors are negative.

$$2x - 1 < 0 \quad \text{and} \quad x - 2 < 0$$

$$x < 0.5 \qquad x < 2$$

Thus, $x < 0.5$.

Therefore, the solution set is $(2, \infty) \cup (-\infty, 0.5)$.

3.
$$4x^2 > 2x + 1$$
$$4x^2 - 2x - 1 > 0$$

The zeros of $4x^2 - 2x - 1$ are $x = \frac{1 \pm \sqrt{5}}{4}$. These values divide the number line into three intervals: $\left(-\infty, \frac{1-\sqrt{5}}{4}\right)$, $\left(\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right)$, and $\left(\frac{1+\sqrt{5}}{4}, \infty\right)$. Try test values for each interval.

For Interval 1, try $x = -1$.

$$f(-1) = 4(-1)^2 - 2(-1) - 1$$

$$> 0$$

For Interval 2, try $x = 0$.

$$f(-1) = 4(0)^2 - 2(0) - 1$$

$$< 0$$

For Interval 3, try $x = 2$.

$$f(-1) = 4(2)^2 - 2(2) - 1 \\ > 0$$

Therefore, the solution set is $\left(-\infty, \frac{1-\sqrt{5}}{4}\right) \cup \left(\frac{1+\sqrt{5}}{4}, \infty\right)$.

Section 1: Follow-up Activities

Extra Help

1. **Case 1:** $x > 0$ and $x - 1 > 0$
Therefore, $x > 1$.

Case 2: $x < 0$ and $x - 1 < 0$
Therefore, $x < 0$.

Thus, the solution set is $(-\infty, 0) \cup (1, \infty)$.

$$\begin{aligned} 2. \quad & \frac{x}{x-1} - 2 \leq 0 \\ & \frac{x}{x-1} - \frac{2(x-1)}{x-1} \leq 0 \\ & \frac{-x+2}{x-1} \leq 0 \end{aligned}$$

Case 1: $-x + 2 \leq 0$ and $x - 1 > 0$
 $x \geq 2 \quad x > 1$

Thus, $x \geq 2$.

Case 2: $-x + 2 \geq 0$ and $x - 1 < 0$
 $x \leq 2 \quad x < 1$

Thus, $x < 1$.

Therefore, the solution set is $[2, \infty) \cup (-\infty, 1)$.

$$\begin{aligned} 3. \quad & \frac{x+1}{x-1} > -2 \\ & \frac{x+1}{x-1} + 2 > 0 \\ & \frac{x+1}{x-1} + \frac{2(x-1)}{x-1} > 0 \\ & \frac{x+1+2x-2}{x-1} > 0 \\ & \frac{3x-1}{x-1} > 0 \end{aligned}$$

Case 1: $3x - 1 > 0$ and $x - 1 > 0$
 $x > \frac{1}{3} \quad x > 1$

Therefore, $x > 1$.

Case 2: $3x - 1 < 0$ and $x - 1 < 0$

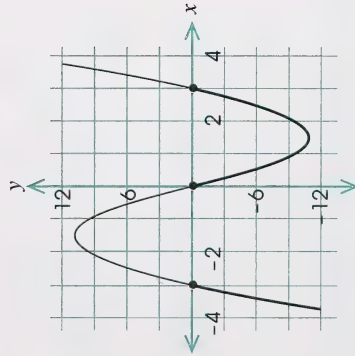
$$x < \frac{1}{3} \quad x < 1$$

Therefore, $x < \frac{1}{3}$.

The solution set is $(-\infty, \frac{1}{3}) \cup (1, \infty)$.

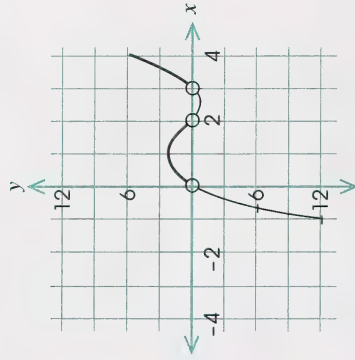
Enrichment

1. $x(x-3)(x+3) \leq 0$



The solution set is $(-\infty, -3] \cup [0, 3] \cup [3, \infty)$.

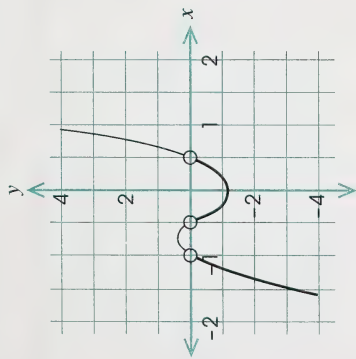
2. $x^3 > 5x^2 - 6x$
 $x^3 - 5x^2 + 6x > 0$



The solution set is $(0, 2) \cup (2, 3)$.

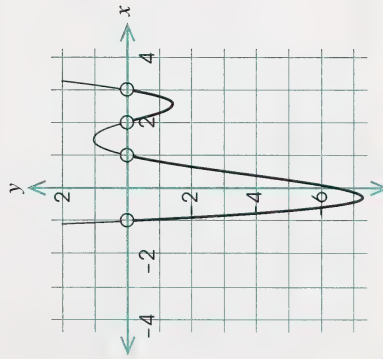
3. $4x^3 + 4x^2 - x < 1$
 $4x^3 + 4x^2 - x - 1 < 0$

Section 2: Activity 1



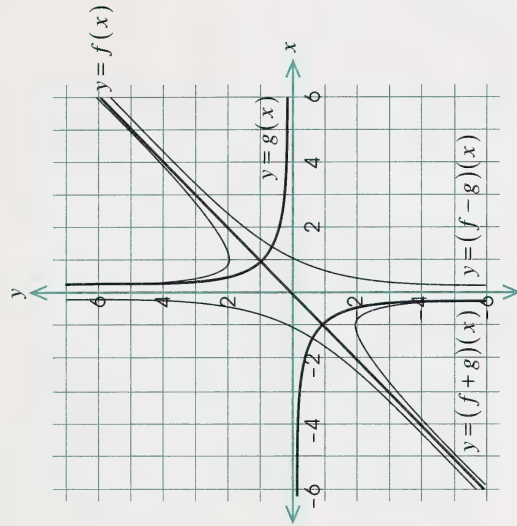
The solution set is $(-\infty, -1) \cup (-\frac{1}{2}, \frac{1}{2})$.

4. $x^4 - 5x^3 + 5x^2 + 5x - 6 < 0$



The solution set is $(-1, 1) \cup (2, 3)$.

1.



2. $(f+g)(x) = x^2 + x - 1$

Domain = Reals

$(f-g)(x) = x^2 - x + 1$

Domain = Reals

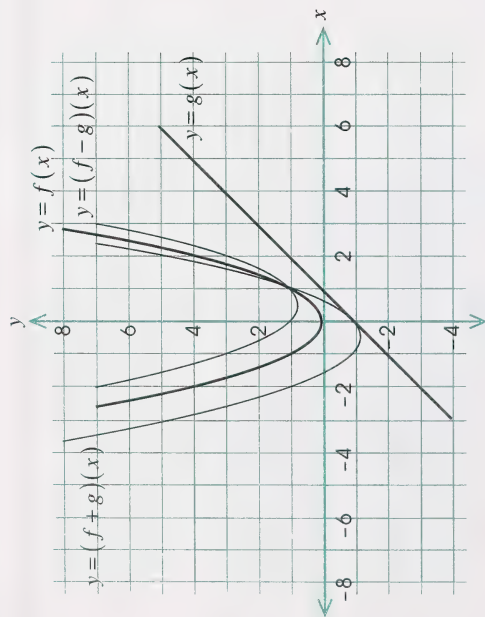
$(fg)(x) = x^3 - x^2$

Domain = Reals

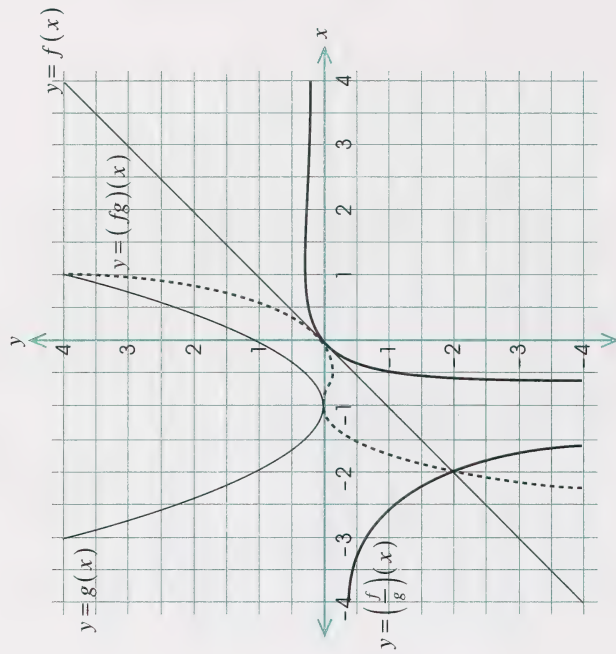
$\left(\frac{f}{g}\right)(x) = \frac{x^2}{x-1}$

Domain = $\{x \mid x \neq 1\}$

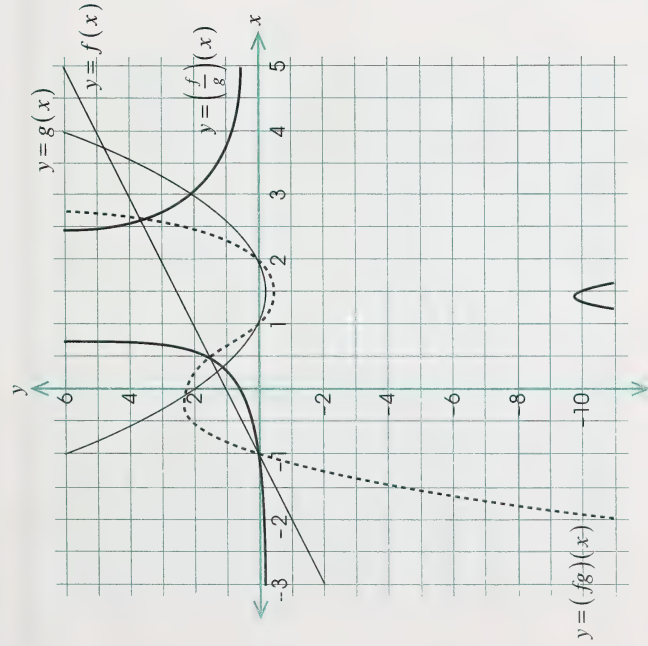
3.



4. Answers will vary. For instance, if $f(x) = x$ and $g(x) = x^2 + 2x + 1$, then $(fg)(x) = x^3 + 2x^2 + x$ and $\left(\frac{f}{g}\right)(x) = \frac{x}{x^2 + 2x + 1}$. The graphs of $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ are shown as follows:



5.



$\frac{f}{g}$ is undefined for $x = 1$ and $x = 2$.

6. $(f + g)(h) = 20 - 0.002h$

$$11 = 20 - 0.002h$$

$$0.002h = 9$$

$$h = 4500$$

The temperature is 11°C at an altitude of 4500 m.

Section 2: Activity 2

1. a. $g(0) = \frac{1}{0-1}$
 $= -1$
 $f(g(0)) = f(-1)$
 $= 1$

b. $f(h(x)) = f(2x-4)$
 $= (2x-4)^2$
 $= 4x^2 - 16x + 16$

c. $g(3) = \frac{1}{3-1}$
 $= \frac{1}{2}$
 $f(g(3)) = f\left(\frac{1}{2}\right)$
 $= \left(\frac{1}{2}\right)^2$
 $= \frac{1}{4}$

$$h(f(g(3))) = h\left(\frac{1}{4}\right)$$

$$= -\frac{7}{2}$$

$$\begin{aligned}
 \text{d. } h(f(g(x))) &= h\left(f\left(\frac{1}{x-1}\right)\right) \\
 &= h\left(\frac{1}{(x-1)^2}\right) \\
 &= 2\left(\frac{1}{(x-1)^2}\right) - 4 \\
 &= \frac{2}{(x-1)^2} + \frac{-4x^2 + 8x - 4}{(x-1)^2} \\
 &= \frac{-4x^2 + 8x - 2}{(x-1)^2}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad g(f(x)) &= g(\sqrt{1-x}) \\
 &= \sqrt{3 - \sqrt{1-x}}
 \end{aligned}$$

Since $f(x)$ is defined for $x \leq 1$, and $y = g(f(x))$ would only be defined if $\sqrt{1-x} \leq 3$ or $1-x \leq 9$ or $x \geq -8$, the domain of $y = g(f(x))$ is $[-8, 1]$.

4. Let A be the area within the circular wave, r be the radius of wave, v be the speed of the wave, and t be the elapsed time.

Since $A = \pi r^2$ and $r = vt$, then the area enclosed by the wave as a function of time is as follows:

$$\begin{aligned}
 A &= \pi v^2 t^2 \\
 &= 4\pi t^2 \text{ m}^2/\text{s}^2 \quad (v = 2 \text{ m/s})
 \end{aligned}$$

$$\begin{aligned}
 2. \quad f(g(x)) &= g(f(x)) \\
 f(x^2) &= g(x+2) \\
 x^2 + 2 &= (x+2)^2 \\
 x^2 + 2 &= x^2 + 4x + 4 \\
 -4x &= 2
 \end{aligned}$$

$$x = -\frac{1}{2}$$

$$\begin{aligned}
 1. \quad (fg)(x) &= (x-1)(2x-1) & f(g(x)) &= f(2x-1) \\
 &= 2x^2 - 3x + 1 & &= 2x-1-1 \\
 & & &= 2x-2
 \end{aligned}$$

Section 2: Follow-up Activities

Extra Help

Section 3: Activity 1

2. $(fg)x = f(g(x))$

$$2x^2 - 3x + 1 = 2x - 2$$

$$2x^2 - 5x + 3 = 0$$

$$(2x - 3)(x - 1) = 0$$

$$\therefore 2x - 3 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = \frac{3}{2} \quad x = 1$$

Therefore, $x = \frac{3}{2}$ or 1.

Enrichment

1. If $g(x) = x + 2$ and $f(x) = x^4$, then $y = f(g(x))$.

2. If $g(x) = x - 7$ and $f(x) = \sqrt{x}$, then $y = f(g(x))$.

3. If $g(x) = x + 4$ and $f(x) = \frac{3}{x}$, then $y = f(g(x))$.

4. If $h(x) = x + 3$, $g(x) = \sqrt{x}$, and $f(x) = x^5$ then $y = f(g(h(x)))$.

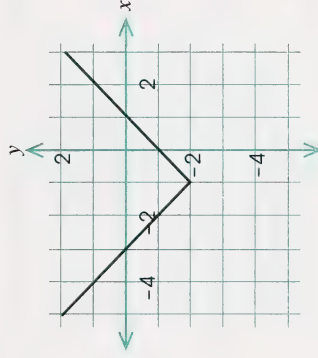
1. If $f(x) = \sqrt{4 - x^2}$, $h(x) = 3$, and $i(x) = x - 3$, then $f_2(x) = f(i(x))$ and $f_4(x) = f(x) - h(x)$.

2. a. $y = \frac{1}{x-3} + 2$
horizontal translation = 3
vertical translation = 2

b. $y = (x + 2)^2 + 1$
horizontal translation = -2
vertical translation = 1

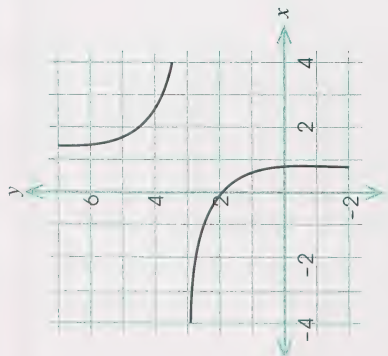
c. $y = x - 2$
horizontal translation = 2 or vertical translation = -2

3. a.

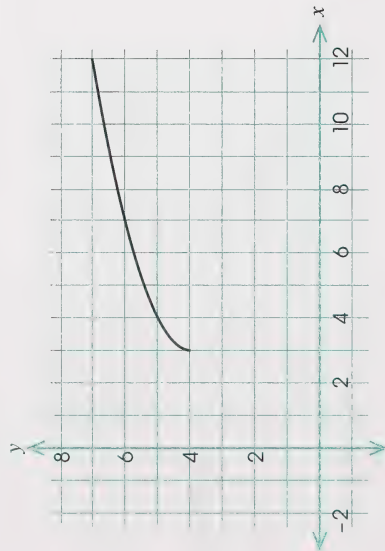


Section 3: Activity 2

b.



c.



4. The equation would be $h = -5t^2 + 4t - 4$.

1. a. reflection in x -axis

$$y = -2^x$$

reflection in y -axis

$$y = 2^{-x}$$

b. reflection in x -axis

$$y = -x$$

reflection in y -axis

$$y = -x$$

c. reflection in x -axis

$$y = -|x|$$

reflection in y -axis

$$y = |x|$$

d. reflection in x -axis

$$y = -x^3$$

reflection in y -axis

$$y = -x^3$$

e. reflection in x -axis

$$y = x^2 + 5x - 1$$

reflection in y -axis

$$y = -x^2 + 5x + 1$$

f. reflection in x -axis

$$y = -x^3 + x$$

reflection in y -axis

$$y = -x^3 + x$$

2. a. $y = -2^{x-2} + 2$

b. $y = (x+3)^2 + 2$ or $y = (-x-3)^2 + 2$

c. $y = -x + 3$

3. The midpoint of PQ is $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ —a point on $y = x$.

The slope of $PQ = \frac{a-b}{b-a} = -1$. The slope of $y = x$ is 1.

Therefore, $y = x$ is perpendicular to PQ , and is the right bisector of PQ .

4. a. $x = y^2$

b. $x = y^2 + 3y$

c. $f^{-1}(x) = x^{\frac{1}{3}}$

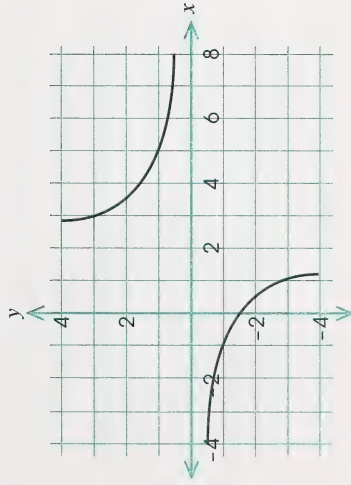
d. $x = |y|$

Section 3: Activity 3

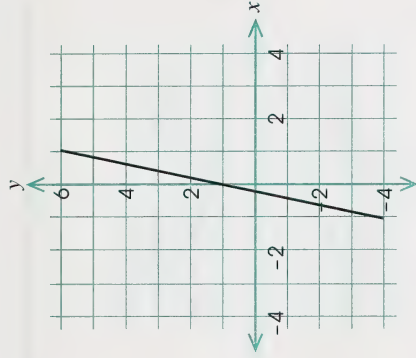
1. Each graph appears to be narrower, because the graph is stretched vertically.

2. $y = -2|x - 2| + 3$

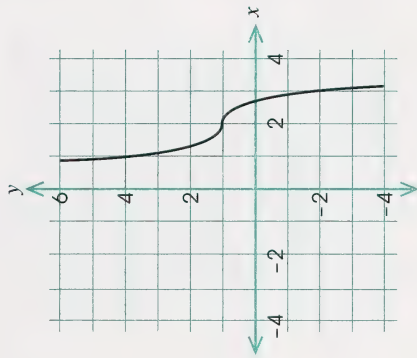
3. a.



b.

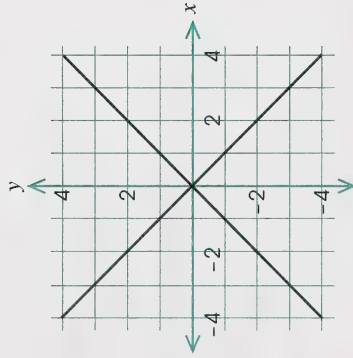


c.

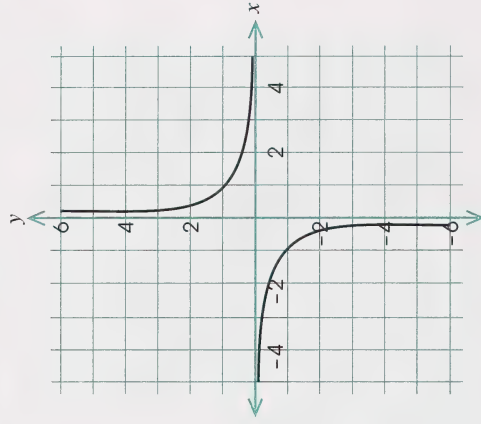


Section 3: Activity 4

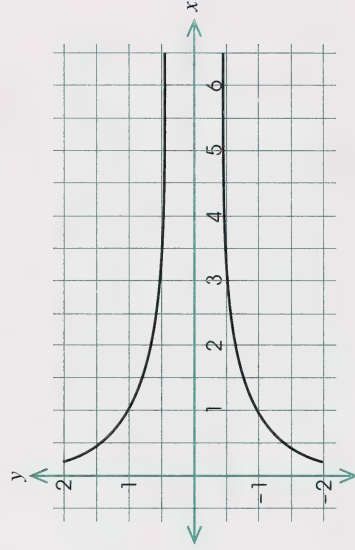
- The graphs of $y = |x|$ and $y = -x^4 - 5x^2$ are symmetric with respect to the y -axis.
- $x = 4$
 - $x = -1$
 - $x = 1$
- The graphs of $y^2 = x - 2$, $x = |y|$, and $y^2 = x^2$ are symmetric with respect to the x -axis.
- $y = 4$
 - $y = 0$
 - $y = 2$
- All graphs are symmetric with respect to the origin.
- $y^2 = x^2$ is symmetric with respect to the x -axis, the y -axis, and the origin.



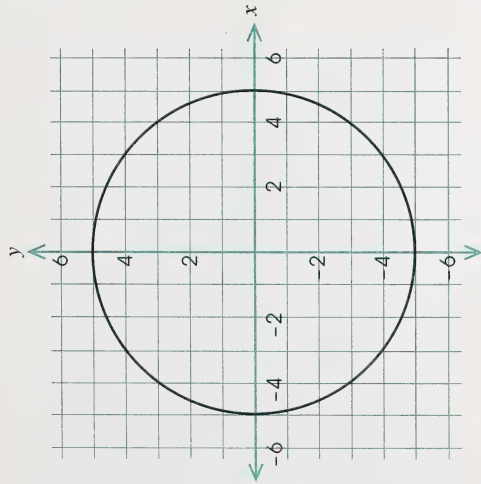
- $y = \frac{1}{x}$ is symmetric with respect to the origin.



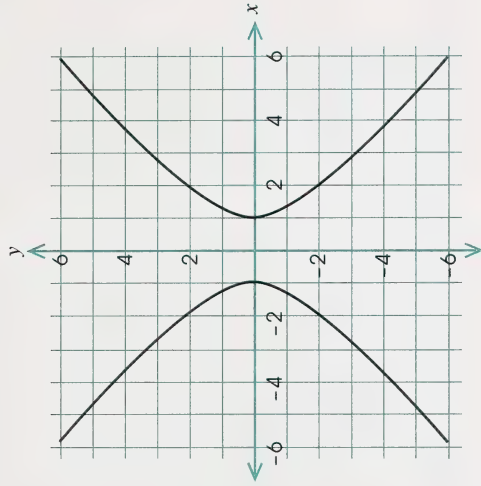
- $y^2 = \frac{1}{x}$ is symmetric with respect to the x -axis.



- d. $x^2 + y^2 = 25$ is symmetric with respect to the x -axis, the y -axis, and the origin.



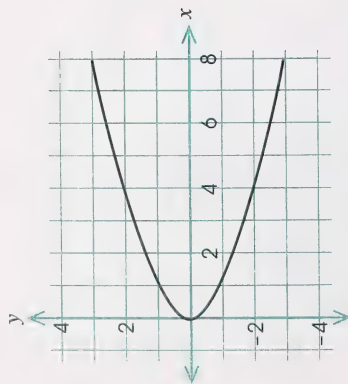
- e. $x^2 - y^2 = 1$ is symmetric with respect to the x -axis, the y -axis, and the origin.



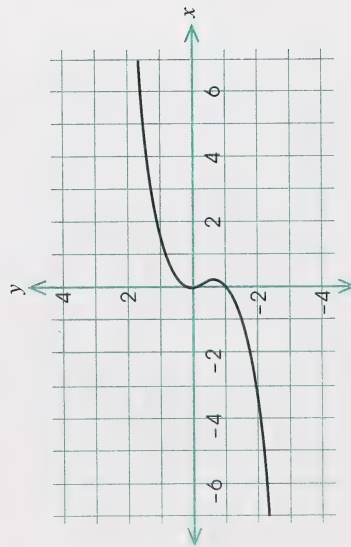
Section 3: Follow-up Activities

Extra Help

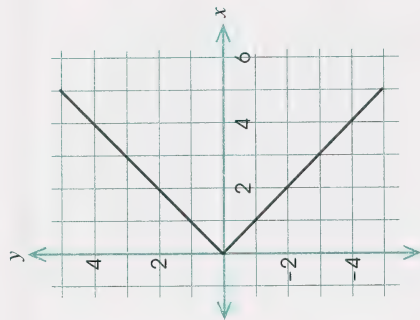
1.



2.



3.



Enrichment

A rotation of 270° produces the same effect as a reflection across $y = x$.

NOTES

NOTES

NOTES



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Mathematics 31

Student Module Booklet

Module 1

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